IWASAWA THEORY AND ZETA ELEMENTS FOR $\mathbb{G}_m$

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Abstract. We describe an explicit 'higher rank' Iwasawa theory for zeta elements associated to the multiplicative group over abelian extensions of general number fields. We then show that this theory leads to a concrete new strategy for proving special cases of the equivariant Tamagawa number conjecture. As a first application of this approach, we use it to prove new cases of the conjecture for Tate motives over natural families of abelian CM-extensions of totally real fields for which the relevant p-adic L-functions possess trivial zeroes.

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1. Introduction

In our previous article [9], we showed that a natural theory of zeta elements associated to the multiplicative group $\mathbb{G}_m$ over finite abelian extensions of number fields shed light on the equivariant Tamagawa number conjecture (or eTNC for short in the remainder of this introduction) in the setting of untwisted Tate motives.

In particular, in this way we derived a wide range of explicit results and predictions concerning, amongst other things, families of fine integral congruence relations between Rubin-Stark elements of different ranks and also several aspects of the detailed Galois module structures of ideal class groups and of the natural Selmer groups (and their homotopy-theoretic transposes) that are associated to $\mathbb{G}_m$. For details see [9].

The main aims of the present article are now to develop an explicit Iwasawa theory for these zeta elements, to use this theory to describe a new approach to proving some important special cases of the eTNC and to describe some initial concrete applications of this approach.

In the next two subsections we discuss briefly the main results that we shall obtain in this direction.

1.1. **Iwasawa main conjectures for general number fields.** The first key aspect of our approach is the formulation of an explicit main conjecture of Iwasawa theory for abelian extensions of *general* number fields (we refer to this conjecture as a 'higher rank main conjecture' since the rank of any associated Euler system would in most cases be greater than one).

To give a little more detail we fix a finite abelian extension K/k of general number fields and a $\mathbb{Z}_p$-extension k_∞ of k and set $K_\infty = Kk_\infty$. In this introduction, we suppose that k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension but this is purely for simplicity.

Our *higher rank main conjecture* is first stated (as Conjecture 3.1, which for pedagogical reasons we refer to as (hIMC) in this introduction) in terms of the existence of an Iwasawa-theoretic zeta element which effectively plays the role of p-adic L-functions for general number fields and has precise prescribed interpolation properties in terms of the values at zero of the higher derivatives of abelian L-series.

We then subsequently reinterpret this conjecture, occasionally under suitable hypotheses, in several more explicit ways: firstly, in terms of the properties of Iwasawa-theoretic 'Rubin-Stark elements' (see Conjecture 3.7), then in terms of the existence of natural Iwasawa-theoretic measures (see Conjecture 3.8), then in terms of the explicit generation, after localization at height one prime ideals, of the higher exterior powers of Iwasawa-theoretic unit groups (see Conjecture 3.13) and finally in a very classical way in terms of the characteristic ideals of suitable ideal class groups and concrete torsion modules constructed from Rubin-Stark elements (see Conjecture 3.19; in fact,

readers who are familiar with the classical formulation of main conjectures may wish to look at this formulation of the conjecture first).

In particular, for the minus part of a CM-abelian extension of a totally real field, we show that (hIMC) is equivalent to the usual main conjecture involving the p-adic L-function of Deligne-Ribet (see the proof of Theorem 3.21(i)) and that for a real abelian field over $\mathbb{Q}$ it is equivalent to the standard formulation of a main conjecture involving cyclotomic units.

In general, the conjecture (hIMC) is implied by the validity of the relevant case of the eTNC for extensions K'/k as K' runs over all (sufficiently large) finite extensions of K in K_∞ but is usually very much weaker than the eTNC (and hence also than the main conjecture formulated by Fukaya and Kato in [16]).

For example, if any p-adic prime of k splits completely in K, then our conjectured zeta element encodes no information concerning the L-values of characters of $\mathrm{Gal}(K/k)$. More precisely, if for any character χ of $\mathrm{Gal}(K/k)$ there exists a p-adic prime of k whose decomposition subgroup in $\mathrm{Gal}(K/k)$ is contained in the kernel of χ (in which case one says that the p-adic L-function for χ has a trivial zero at $s = 0$), then the zeta element that we predict to exist has no conjectured interpolation property involving the leading term of $L_{k,S,T}(\chi, s)$ at $s = 0$.

1.2. **eTNC and congruences between Rubin-Stark elements.** We now turn to discuss how (hIMC) leads to a concrete strategy to prove some interesting new cases of the eTNC.

Here, a key role is played by a detailed Iwasawa-theoretic study of the fine congruence relations between Rubin-Stark elements of differing ranks that were independently formulated in the context of finite abelian extensions by Mazur and Rubin in [24] (where the congruences are referred to as a 'refined class number formula for $\mathbb{G}_m$') and by the third author in [28]. In particular, working in the setting of the extension K_∞/k we formulate an explicit conjecture, denoted for convenience (MRS) here, which, roughly speaking, describes the precise relation between the natural Rubin-Stark elements for K_∞/k and for K/k. (For full details see Conjectures 4.1 and 4.2).

To better understand the context of this conjecture we prove in Theorem 4.9 that it constitutes a natural generalization of the so-called 'Gross-Stark conjecture' formulated by Gross in [19].

It is easy to see that, as already observed above, the eTNC implies the validity of (hIMC) and, in addition, one of the main results of our previous work [9] allows us to prove in a straightforward way that it also implies the validity of (MRS).

One of the key observations of the present article is that, much more significantly, one can prove under certain natural hypotheses a powerful converse to these implications. To be a little more precise, we shall prove a result of the following sort (for a detailed statement of which see Theorem 5.2).

Theorem 1.1. *If the Galois coinvariants of a certain natural Iwasawa module is finite (as has been conjectured to be the case by Gross), then the validity of both (hIMC) and (MRS) for the extension K_∞/k combine to imply the validity of the p-component of the eTNC for every finite subextension F/k of K_∞/k.*

To give a first indication of the usefulness of the above theorem, we apply it in the case that k is totally real and K is CM and consider the 'minus component' of the p-part of the eTNC. In this context we write K^+ for the maximal totally real subfield of K.

We recall that if no p-adic place splits in K/K^+ and the Iwasawa-theoretic μ-invariant of K_∞/K vanishes, then the validity of the minus component of the p-part of the eTNC is already known (as far as we are aware, such a result was first implicitly discussed in the survey article of Flach [14]).

However, by combining Theorems 4.9 and 5.2 with recent work of Darmon, Dasgupta and Pollack [12] and of Ventullo [35] on the Gross-Stark conjecture, we can now prove the following result (for a precise statement of which see Corollary 5.8).

Corollary 1.2. *Let K be a finite CM-extension of a totally real field k. Let p be an odd prime for which the Iwasawa-theoretic μ-invariant of K_∞/K vanishes and at most one p-adic place of k splits in K/K^+. Then the minus component of the p-part of the eTNC for K/k is (unconditionally) valid.*

We remark that this result gives *the first verifications* of the (minus component of the p-part of the) eTNC for the untwisted Tate motive over abelian CM-extensions of a totally real field that is not equal to $\mathbb{Q}$ and for which *the relevant p-adic L-series possess trivial zeroes.* For details of some concrete applications of Corollary 1.2 in this regard, see Examples 5.9.

In another direction, Corollary 1.2 also leads directly to a strong refinement of one of the main results of Greither and Popescu in [18] (for details of which see Corollary 3.27).

1.3. **Further developments.** Finally we would like to point out that the ideas presented in this article extend naturally in at least two different directions and that we intend to discuss these developments elsewhere.

Firstly, one can formulate a natural generalization of the theory discussed here in the context of arbitrary Tate motives. In this setting our theory is related to natural generalizations of both the notion of Rubin-Stark element (which specializes in the case of Tate motives of strictly positive weight over abelian extensions of $\mathbb{Q}$ to recover Soulé's construction of cyclotomic elements in higher algebraic K-theory) and of the Rubin-Stark conjecture itself. In particular, our approach leads in this context to the formulation of precise conjectural congruence relations between Rubin-Stark elements of differing 'weights' which can be seen to constitute a wide-ranging (conjectural) generalization of the classical Kummer congruences involving Bernoulli numbers. For more details in this regard see §3.2.2.

Secondly, many of the constructions, conjectures and results that are discussed here extend naturally to the setting of non-commutative Iwasawa theory and can then be used to prove the same case of the eTNC that we consider here over natural families of Galois extensions that are both non-abelian and of degree divisible by a prime p at which the relevant p-adic L-series possess trivial zeroes.

Notation. For the reader's convenience we start by collecting some basic notation.

For any (profinite) group G we write $\widehat{G}$ for the group of homomorphisms $G \to \mathbb{C}^\times$ of finite order.

Let k be a number field. For a place v of k, the residue field of v is denoted by $\kappa(v)$, and its order is denoted by $\mathrm{N}v$. We denote the set of places of k which lie above the infinite place ∞ of $\mathbb{Q}$ (resp. a prime number p) by $S_\infty(k)$ (resp. $S_p(k)$). For a Galois extension L/k, the set of places of k which ramify in L is denoted by $S_{\mathrm{ram}}(L/k)$. For any set Σ of places of k, we denote by Σ_L the set of places of L which lie above places in Σ.

Let L/k be an abelian extension with Galois group G. For a place v of k, the decomposition group at v in G is denoted by G_v. If v is unramified in L, the Frobenius automorphism at v is denoted by Fr_v.

Let E be either a field of characteristic 0 or $\mathbb{Z}_p$. For an abelian group A, we denote $E \otimes_{\mathbb{Z}} A$ by EA or A_E. For a $\mathbb{Z}_p$-module A and an extension field E of $\mathbb{Q}_p$, we also write EA or A_E for $E \otimes_{\mathbb{Z}_p} A$. (This abuse of notation would not make any confusion.) We use similar notation for complexes. For example, if C is a complex of abelian groups, then we denote $E \otimes^{\mathbb{L}}_{\mathbb{Z}} C$ by EC or C_E.

Let R be a commutative ring, and M be an R-module. Let r and s be non-negative integers with $r \leq s$. There is a canonical paring

$$\bigwedge_R^s M \times \bigwedge_R^r \mathrm{Hom}_R(M, R) \to \bigwedge_R^{s-r} M$$

defined by

$$(a_1 \wedge \cdots \wedge a_s, \varphi_1 \wedge \cdots \wedge \varphi_r) \mapsto \sum_{\sigma \in \mathfrak{S}_{s,r}} \mathrm{sgn}(\sigma) \det(\varphi_i(a_{\sigma(j)}))_{1 \leq i,j \leq r} a_{\sigma(r+1)} \wedge \cdots \wedge a_{\sigma(s)},$$

where

$$\mathfrak{S}_{s,r} := \{\sigma \in \mathfrak{S}_s \mid \sigma(1) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \cdots < \sigma(s)\}.$$

(See [9, Proposition 4.1].) We denote the image of (a, Φ) under the above pairing by $\Phi(a)$.

For any R-module M, we denote the linear dual $\mathrm{Hom}_R(M, R)$ by M^*.

The total quotient ring of R is denoted by $Q(R)$.

2. Zeta elements for $\mathbb{G}_m$

In this section, we review the zeta elements for $\mathbb{G}_m$ that were studied in [9].

2.1. **The Rubin-Stark conjecture.** We review the formulation of the Rubin-Stark conjecture [27, Conjecture B′].

Let L/k be a finite abelian extension of number fields with Galois group G. Let S be a finite set of places of k which contains $S_\infty(k) \cup S_{\rm ram}(L/k)$. We fix a labeling $S = \{v_0, \ldots, v_n\}$. Take $r \in \mathbb{Z}$ so that $v_1, \ldots, v_r$ split completely in L. We put $V := \{v_1, \ldots, v_r\}$. For each place v of k, we fix a place w of L lying above v. In particular, for each i with $0 \leq i \leq n$, we fix a place w_i of L lying above v_i. Such conventions are frequently used in this paper.

For $\chi \in \widehat{G}$, let $L_{k,S}(\chi, s)$ denote the usual S-truncated L-function for χ. We put

$$r_{\chi,S} := {\rm ord}_{s=0} L_{k,S}(\chi, s).$$

Let $\mathcal{O}_{L,S}$ be the ring of S_L integers of L. For any set Σ of places of k, put $Y_{L,\Sigma} := \bigoplus_{w\in\Sigma_L} \mathbb{Z}w$, the free abelian group on Σ_L. We define

$$X_{L,\Sigma} := \{\sum_{w\in\Sigma_L} a_w w \in Y_{L,\Sigma} \mid \sum_{w\in\Sigma_L} a_w = 0\}.$$

By Dirichlet's unit theorem, we know that the homomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{L,S} : \mathbb{R}\mathcal{O}_{L,S}^\times \xrightarrow{\sim} \mathbb{R}X_{L,S};\ a \mapsto -\sum_{w\in S_L} \log |a|_w w$$

is an isomorphism.

By [34, Chap. I, Proposition 3.4] we know that

$$\begin{aligned} r_{\chi,S} &= \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S}^\times) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}X_{L,S}) \\ &= \begin{cases} \#\{v \in S \mid \chi(G_v) = 1\} & \text{if } \chi \neq 1, \\ n(= \#S - 1) & \text{if } \chi = 1, \end{cases} \end{aligned}$$

where $e_\chi := \frac{1}{\#G}\sum_{\sigma\in G} \chi(\sigma)\sigma^{-1}$. From this fact, we see that $r \leq r_{\chi,S}$.

Let T be a finite set of places of k which is disjoint from S. The S-truncated T-modified L-function is defined by

$$L_{k,S,T}(\chi, s) := (\prod_{v\in T}(1 - \chi({\rm Fr}_v){\rm N}v^{1-s}))L_{k,S}(\chi, s).$$

The (S,T)-unit group of L is defined by

$$\mathcal{O}_{L,S,T}^\times := \ker(\mathcal{O}_{L,S}^\times \to \bigoplus_{w\in T_L} \kappa(w)^\times).$$

Note that $\mathcal{O}_{L,S,T}^\times$ is a subgroup of $\mathcal{O}_{L,S}^\times$ of finite index. We have

$$r \leq r_{\chi,S} = {\rm ord}_{s=0} L_{k,S,T}(\chi, s) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S,T}^\times).$$

We put

$$L_{k,S,T}^{(r)}(\chi, 0) := \lim_{s\to 0} s^{-r} L_{k,S,T}(\chi, s).$$

We define the r-th order Stickelberger element by

$$\theta_{L/k,S,T}^{(r)} := \sum_{\chi \in \widehat{G}} L_{k,S,T}^{(r)}(\chi^{-1}, 0) e_\chi \in \mathbb{R}[G].$$

The (r-th order) Rubin-Stark element

$$\epsilon_{L/k,S,T}^{V} \in \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times}$$

is defined to be the element which corresponds to

$$\theta_{L/k,S,T}^{(r)} \cdot (w_1 - w_0) \wedge \cdots \wedge (w_r - w_0) \in \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S}$$

under the isomorphism

$$\mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \xrightarrow{\sim} \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S}$$

induced by $\lambda_{L,S}$.

Now assume that $\mathcal{O}_{L,S,T}^{\times}$ is $\mathbb{Z}$-free. Then, the Rubin-Stark conjecture predicts that the Rubin-Stark element $\epsilon_{L/k,S,T}^{V}$ lies in the lattice

$$\bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} := \{a \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \mid \Phi(a) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S,T}^{\times}, \mathbb{Z}[G])\}.$$

(See [27, Conjecture B′].) In this paper, we consider the 'p-part' of the Rubin-Stark conjecture for a fixed prime number p. We put

$$U_{L,S,T} := \mathbb{Z}_p \mathcal{O}_{L,S,T}^{\times}.$$

We also fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. From this, we regard

$$\epsilon_{L/k,S,T}^{V} \in \mathbb{C}_p \bigwedge_{\mathbb{Z}_p[G]}^{r} U_{L,S,T}.$$

We define

$$\bigcap_{\mathbb{Z}_p[G]}^{r} U_{L,S,T} := \{a \in \mathbb{Q}_p \bigwedge_{\mathbb{Z}_p[G]}^{r} U_{L,S,T} \mid \Phi(a) \in \mathbb{Z}_p[G] \text{ for all } \Phi \in \bigwedge_{\mathbb{Z}_p[G]}^{r} \mathrm{Hom}_{\mathbb{Z}_p[G]}(U_{L,S,T}, \mathbb{Z}_p[G])\}.$$

We easily see that there is a natural isomorphism $\mathbb{Z}_p \bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \simeq \bigcap_{\mathbb{Z}_p[G]}^{r} U_{L,S,T}$. We often denote $\bigwedge_{\mathbb{Z}_p[G]}^{r}$ and $\bigcap_{\mathbb{Z}_p[G]}^{r}$ simply by $\bigwedge^{r}$ and $\bigcap^{r}$ respectively.

We propose the 'p-component version' of the Rubin-Stark conjecture as follows.

Conjecture 2.1 ($\mathrm{RS}(L/k,S,T,V)_p$).

$$\epsilon^V_{L/k,S,T} \in \bigcap^r U_{L,S,T}.$$

Remark 2.2. Concerning known results on the Rubin-Stark conjecture, see [9, Remark 5.3] for example. Note that the Rubin-Stark conjecture is a consequence of the eTNC. This result was first proved by the first author in [3, Corollary 4.1], and later by the present authors [9, Theorem 5.13] in a simpler way.

2.2. **The eTNC for the untwisted Tate motive.** In this subsection, we review the formulation of the eTNC for the untwisted Tate motive.

Let $L/k, G, S, T$ be as in the previous subsection. Fix a prime number p. We assume that $S_p(k) \subset S$. Consider the complex

$$C_{L,S} := R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p),\mathbb{Z}_p)[-2].$$

It is known that $C_{L,S}$ is a perfect complex of $\mathbb{Z}_p[G]$-modules, acyclic outside degrees zero and one. We have a canonical isomorphism

$$H^0(C_{L,S}) \simeq U_{L,S}(:= \mathbb{Z}_p\mathcal{O}^\times_{L,S}),$$

and a canonical exact sequence

$$0 \to A_S(L) \to H^1(C_{L,S}) \to \mathcal{X}_{L,S} \to 0,$$

where $A_S(L) := \mathbb{Z}_p \operatorname{Pic}(\mathcal{O}_{L,S})$ and $\mathcal{X}_{L,S} := \mathbb{Z}_p X_{L,S}$. The complex $C_{L,S}$ is identified with the p-completion of the complex obtained from the classical 'Tate sequence' (if S is large enough), and also identified with $\mathbb{Z}_p R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{G}_m)$, where $R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{G}_m)$ is the 'Weil-étale cohomology complex' constructed in [9, §2.2] (see [5, Proposition 3.3] and [4, Proposition 3.5(e)]).

By a similar construction with [9, Proposition 2.4], we construct a canonical complex $C_{L,S,T}$ which lies in the distinguished triangle

$$C_{L,S,T} \to C_{L,S} \to \bigoplus_{w\in T_L} \mathbb{Z}_p\kappa(w)^\times[0].$$

(Simply we can define $C_{L,S,T}$ by $\mathbb{Z}_p R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{G}_m)$ in the terminology of [9].) We have

$$H^0(C_{L,S,T}) = U_{L,S,T}$$

and the exact sequence

$$0 \to A^T_S(L) \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S} \to 0,$$

where $A^T_S(L)$ is the p-part of the ray class group of $\mathcal{O}_{L,S}$ with modulus $\prod_{w\in T_L} w$.

We define the leading term of $L_{k,S,T}(\chi,s)$ at $s=0$ by

$$L^*_{k,S,T}(\chi,0) := \lim_{s\to 0} s^{-r_{\chi,S}} L_{k,S,T}(\chi,s).$$

The leading term at $s = 0$ of the equivariant L-function

$$\theta_{L/k,S,T}(s) := \sum_{\chi\in\widehat{G}} L_{k,S,T}(\chi^{-1}, s)e_\chi$$

is defined by

$$\theta^*_{L/k,S,T}(0) := \sum_{\chi\in\widehat{G}} L^*_{k,S,T}(\chi^{-1}, 0)e_\chi \in \mathbb{R}[G]^\times.$$

As in the previous subsection we fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. We regard $\theta^*_{L/k,S,T}(0) \in \mathbb{C}_p[G]^\times$. The zeta element for $\mathbb{G}_m$

$$z_{L/k,S,T} \in \mathbb{C}_p\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T})$$

is defined to be the element which corresponds to $\theta^*_{L/k,S,T}(0)$ under the isomorphism

$$\begin{aligned} \mathbb{C}_p\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\simeq \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}) \\ &\stackrel{\sim}{\to} \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}) \\ &\stackrel{\sim}{\to} \mathbb{C}_p[G], \end{aligned}$$

where the second isomorphism is induced by $\lambda_{L,S}$, and the last isomorphism is the evaluation map. Note that determinant modules must be regarded as graded invertible modules, but we omit the grading of any graded invertible modules as in [9].

The eTNC for the pair $(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is formulated as follows.

Conjecture 2.3 (eTNC($h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]$))**.**

$$\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}).$$

Remark 2.4. When p is odd, k is totally real, and L is CM, we say that the minus part of the eTNC (which we denote by eTNC($h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-$)) is valid if we have the equality

$$e^-\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = e^-\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}),$$

where $e^- := \frac{1-c}{2}$ and $c \in G$ is the complex conjugation.

2.3. **The eTNC and Rubin-Stark elements.** In this subsection, we interpret the eTNC, using Rubin-Stark elements. The result in this subsection will be used in §5.

We continue to use the notation in the previous subsection. Take $\chi \in \widehat{G}$, and suppose that $r_{\chi,S} < \#S$. Put $L_\chi := L^{\ker\chi}$ and $G_\chi := \mathrm{Gal}(L_\chi/k)$. Take $V_{\chi,S} \subset S$ so that all $v \in V_{\chi,S}$ split completely in L_χ (i.e. $\chi(G_v) = 1$) and $\#V_{\chi,S} = r_{\chi,S}$. Note that, if $\chi \neq 1$, we have

$$V_{\chi,S} = \{v \in S \mid \chi(G_v) = 1\}.$$

Consider the Rubin-Stark element

$$\epsilon^{V_{\chi,S}}_{L_\chi/k,S,T} \in \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T}.$$

Note that a Rubin-Stark element depends on a fixed labeling of S, so in this case a labeling of S such that $S=\{v_0,\ldots,v_n\}$ and $V_{\chi,S}=\{v_1,\ldots,v_{r_{\chi,S}}\}$ is understood to be chosen.

For a set Σ of places of k and a finite extension F/k, put $\mathcal{Y}_{F,\Sigma}:=\mathbb{Z}_pY_{F,\Sigma}=\bigoplus_{w\in\Sigma_F}\mathbb{Z}_pw$ and $\mathcal{X}_{F,\Sigma}:=\mathbb{Z}_pX_{F,\Sigma}=\ker(\mathcal{Y}_{F,\Sigma}\to\mathbb{Z}_p)$. The natural surjection

$$\mathcal{X}_{L_\chi,S}\to\mathcal{Y}_{L_\chi,V_{\chi,S}}$$

induces an injection

$$\mathcal{Y}^*_{L_\chi,V_{\chi,S}}\to\mathcal{X}^*_{L_\chi,S},$$

where $(\cdot)^*:=\mathrm{Hom}_{\mathbb{Z}_p[G_\chi]}(\cdot,\mathbb{Z}_p[G_\chi])$. Since $\mathcal{Y}_{L_\chi,V_{\chi,S}}\simeq\mathbb{Z}_p[G_\chi]^{\oplus r_{\chi,S}}$ and $\dim_{\mathbb{C}_p}(e_\chi\mathbb{C}_p\mathcal{X}_{L,S})=r_{\chi,S}$, the above map induces an isomorphism

$$e_\chi\mathbb{C}_p\mathcal{Y}^*_{L_\chi,V_{\chi,S}}\xrightarrow{\sim}e_\chi\mathbb{C}_p\mathcal{X}^*_{L,S}.$$

From this, we have a canonical identification

$$e_\chi\mathbb{C}_p(\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}\otimes\bigwedge^{r_{\chi,S}}\mathcal{Y}^*_{L_\chi,V_{\chi,S}})=e_\chi(\mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T})\otimes_{\mathbb{C}_p[G]}\mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S})).$$

Since $\{w_1,\ldots,w_{r_{\chi,S}}\}$ is a basis of $\mathcal{Y}_{L_\chi,V_{\chi,S}}$, we have the (non-canonical) isomorphism

$$\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}\xrightarrow{\sim}\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}\otimes\bigwedge^{r_{\chi,S}}\mathcal{Y}^*_{L_\chi,V_{\chi,S}};\ a\mapsto a\otimes w_1^*\wedge\cdots\wedge w^*_{r_{\chi,S}},$$

where w_i^* is the dual of w_i. Hence, we have the (non-canonical) isomorphism

$$e_\chi\mathbb{C}_p\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}\simeq e_\chi(\mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T})\otimes_{\mathbb{C}_p[G]}\mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S})).$$

Proposition 2.5. *Suppose that $r_{\chi,S}<\#S$ for every $\chi\in\widehat{G}$. Then,* $\mathrm{eTNC}(h^0(\mathrm{Spec}\,L),\mathbb{Z}_p[G])$ *holds if and only if there exists a $\mathbb{Z}_p[G]$-basis $\mathcal{L}_{L/k,S,T}$ of $\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T})$ such that, for every $\chi\in\widehat{G}$, the image of $e_\chi\mathcal{L}_{L/k,S,T}$ under the isomorphism*

$$e_\chi\mathbb{C}_p\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T})\simeq e_\chi(\mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T})\otimes_{\mathbb{C}_p[G]}\mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}))\simeq e_\chi\mathbb{C}_p\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}$$

coincides with $e_\chi\epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$.

Proof. By the definition of Rubin-Stark elements, we see that the image of $e_\chi\epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$ under the isomorphism

$$\begin{aligned}e_\chi\mathbb{C}_p\bigwedge^{r_{\chi,S}}U_{L_\chi,S,T}&\simeq e_\chi(\mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T})\otimes_{\mathbb{C}_p[G]}\mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}))\\&\simeq e_\chi(\mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S})\otimes_{\mathbb{C}_p[G]}\mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p\mathcal{X}_{L,S}))\\&\simeq e_\chi\mathbb{C}_p[G]\end{aligned}$$

is equal to $e_\chi L^*_{k,S,T}(\chi^{-1},0)$. The 'only if part' follows by putting $\mathcal{L}_{L/k,S,T}:=z_{L/k,S,T}$. The 'if part' follows by noting that $\mathcal{L}_{L/k,S,T}$ must be equal to $z_{L/k,S,T}$. □

2.4. **The canonical projection maps.** Let $L/k, G, S, T, V, r$ be as in §2.1. We put

$$e_r := \sum_{\chi \in \widehat{G},\ r_{\chi,S}=r} e_\chi \in \mathbb{Q}[G].$$

As in Proposition 2.5, we construct the (non-canonical) isomorphism

$$e_r\mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) \simeq e_r\mathbb{C}_p \bigwedge^r U_{L,S,T}.$$

In this subsection, we give an explicit description of the map

$$\pi^V_{L/k,S,T} : \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) \overset{e_r\mathbb{C}_p\otimes}{\to} e_r\mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) \simeq e_r\mathbb{C}_p \bigwedge^r U_{L,S,T} \subset \mathbb{C}_p \bigwedge^r U_{L,S,T}.$$

This map is important since the image of the zeta element $z_{L/k,S,T}$ under this map is the Rubin-Stark element $\epsilon^V_{L/k,S,T}$.

Firstly, we choose a representative of $C_{L,S,T}$

$$\Pi \overset{\psi}{\to} \Pi,$$

where the first term is placed in degree zero, such that Π is a free $\mathbb{Z}_p[G]$-module with basis $\{b_1, \dots, b_d\}$ (d is sufficiently large), and that the natural surjection

$$\Pi \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S}$$

sends b_i to $w_i - w_0$ for each i with $1 \le i \le r$. For the details of this construction, see [9, §5.4]. Note that the representative of $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ chosen in [9, §5.4] is of the form

$$P \to F,$$

where P is projective and F is free. By Swan's theorem [11, (32.1)], we have an isomorphism $\mathbb{Z}_pP \simeq \mathbb{Z}_pF$. This shows that we can take the representative of $C_{L,S,T}$ as above.

We define $\psi_i \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ by

$$\psi_i := b_i^* \circ \psi,$$

where b_i^* is the dual of b_i. Note that $\bigwedge_{r<i\le d} \psi_i \in \bigwedge^{d-r} \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ defines the homomorphism

$$\bigwedge_{r<i\le d} \psi_i : \bigwedge^d \Pi \to \bigwedge^r \Pi$$

given by

$$(\bigwedge_{r<i\le d} \psi_i)(b_1 \wedge \cdots \wedge b_d) = \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}$$

(see Notation.)

Proposition 2.6.

(i) *We have*
$$\bigcap^r U_{L,S,T} = (\mathbb{Q}_p \bigwedge^r U_{L,S,T}) \cap \bigwedge^r \Pi,$$
where we regard $U_{L,S,T} \subset \Pi$ *via the natural inclusion*
$$U_{L,S,T} = H^0(C_{L,S,T}) = \ker \psi \hookrightarrow \Pi.$$

(ii) *If we regard* $\bigcap^r U_{L,S,T} \subset \bigwedge^r \Pi$ *by (i), then we have*
$$\mathrm{im}(\bigwedge_{r<i\le d} \psi_i : \bigwedge^d \Pi \to \bigwedge^r \Pi) \subset \bigcap^r U_{L,S,T}.$$

(iii) *The map*
$$\det_{\mathbb{Z}_p[G]}(C_{L,S,T}) = \bigwedge^d \Pi \otimes \bigwedge^d \Pi^* \to \bigcap^r U_{L,S,T};\ b_1\wedge\cdots\wedge b_d \otimes b_1^* \wedge\cdots\wedge b_d^* \mapsto (\bigwedge_{r<i\le d} \psi_i)(b_1\wedge\cdots\wedge b_d)$$
coincides with $(-1)^{r(d-r)}\pi^V_{L/k,S,T}$*. In particular, we have*
$$\pi^V_{L/k,S,T}(b_1\wedge\cdots\wedge b_d\otimes b_1^*\wedge\cdots\wedge b_d^*) = (-1)^{r(d-r)} \sum_{\sigma\in\mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma)\det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} b_{\sigma(1)}\wedge\cdots\wedge b_{\sigma(r)}$$
and
$$\mathrm{im}\,\pi^V_{L/k,S,T} \subset \{a \in \bigcap^r U_{L,S,T} \mid e_r a = a\}.$$

Proof. For (i), see [9, Lemma 4.7(ii)]. For (ii) and (iii), see [9, Lemma 4.3]. □

3. Higher rank Iwasawa theory

3.1. **Notation.** We fix a prime number p. We use the following notation:

- k: number field;
- K_∞/k: Galois extension such that $\mathcal{G} := \mathrm{Gal}(K_\infty/k) \simeq \Delta \times \Gamma$, where Δ is a finite abelian group and $\Gamma \simeq \mathbb{Z}_p$;
- $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$;
- Fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, and identify $\widehat{\Delta}$ with $\mathrm{Hom}_{\mathbb{Z}}(\Delta, \overline{\mathbb{Q}}_p^\times)$. For $\chi \in \widehat{\Delta}$, put $\Lambda_\chi := \mathbb{Z}_p[\mathrm{im}\,\chi][[\Gamma]]$.

Note that the total quotient ring $Q(\Lambda)$ has the decomposition
$$Q(\Lambda) \simeq \bigoplus_{\chi\in\widehat{\Delta}/\sim_{\mathbb{Q}_p}} Q(\Lambda_\chi),$$
where the equivalence relation $\sim_{\mathbb{Q}_p}$ is defined by
$$\chi \sim_{\mathbb{Q}_p} \chi' \Leftrightarrow \text{there exists } \sigma \in G_{\mathbb{Q}_p} \text{ such that } \chi = \sigma\circ\chi'.$$

- $K := K_\infty^\Gamma$ (so $\mathrm{Gal}(K/k) = \Delta$);
- $k_\infty := K_\infty^\Delta$ (so k_∞/k is a $\mathbb{Z}_p$-extension with Galois group Γ);

- k_n: the n-th layer of k_∞/k;
- K_n: the n-th layer of K_∞/K;
- $\mathcal{G}_n := \mathrm{Gal}(K_n/k)$.

For each character $\chi \in \widehat{\mathcal{G}}$ we also set

- $L_\chi := K_\infty^{\ker\chi}$;
- $L_{\chi,\infty} := L_\chi \cdot k_\infty$;
- $L_{\chi,n}$: the n-th layer of $L_{\chi,\infty}/L_\chi$;
- $\mathcal{G}_\chi := \mathrm{Gal}(L_{\chi,\infty}/k)$;
- $\mathcal{G}_{\chi,n} := \mathrm{Gal}(L_{\chi,n}/k)$;
- $G_\chi := \mathrm{Gal}(L_\chi/k)$;
- $\Gamma_\chi := \mathrm{Gal}(L_{\chi,\infty}/L_\chi)$;
- $\Gamma_{\chi,n} := \mathrm{Gal}(L_{\chi,n}/L_\chi)$;
- S: a finite set of places of k which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$;
- T: a finite set of places of k which is disjoint from S;
- $V_\chi := \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$ (this is a proper subset of S);
- $r_\chi := \# V_\chi$.

For any intermediate field L of K_∞/k, we denote $\varprojlim_F U_{F,S,T}$ by $U_{L,S,T}$, where F runs over all intermediate field of L/k which is finite over k and the inverse limit is taken with respect to norm maps. Similarly, $C_{L,S,T}$ is defined to be the inverse limit of $C_{F,S,T}$. We denote $\varprojlim_F \mathcal{Y}_{F,S}$ by $\mathcal{Y}_{L,S}$, where the inverse limit is taken with respect to the maps

$$\mathcal{Y}_{F',S} \to \mathcal{Y}_{F,S};\ w_{F'} \mapsto w_F,$$

where $F \subset F'$, $w_{F'} \in S_{F'}$, and $w_F \in S_F$ is the place lying under $w_{F'}$. We use similar notation for $\mathcal{X}_{L,S}$ etc.

3.2. **Iwasawa main conjecture I.** In this section we formulate the main conjecture of Iwasawa theory for general number fields, that is a key to our study.

3.2.1. For any character χ in $\widehat{\mathcal{G}}$ there is a natural composite homomorphism

$$\begin{aligned}
\lambda_\chi : \det{}_\Lambda(C_{K_\infty,S,T}) &\to \det{}_{\mathbb{Z}_p[G_\chi]}(C_{L_\chi,S,T}) \\
&\hookrightarrow \det{}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p C_{L_\chi,S,T}) \\
&\xrightarrow{\sim} \det{}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p U_{L_\chi,S,T}) \otimes_{\mathbb{C}_p[G_\chi]} \det{}^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \\
&\xrightarrow{\sim} \det{}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \otimes_{\mathbb{C}_p[G_\chi]} \det{}^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \\
&\simeq \mathbb{C}_p[G_\chi] \\
&\xrightarrow{\chi} \mathbb{C}_p,
\end{aligned}$$

where the fourth map is induced by $\lambda_{L_\chi,S}$, the fifth map is the evaluation, and the last map is induced by χ.

We can now state our higher rank main conjecture of Iwasawa theory in its first form.

Conjecture 3.1 (IMC$(K_\infty/k,S,T,p)$). *There exists a Λ-basis $\mathcal{L}_{K_\infty/k,S,T}$ of the module $\det_\Lambda(C_{K_\infty,S,T})$ for which, at every $\chi\in\widehat{\Delta}$ and every $\psi\in\widehat{\mathcal{G}_\chi}$ for which $r_{\psi,S}=r_\chi$ one has $\lambda_\psi(\mathcal{L}_{K_\infty/k,S,T})=L^{(r_\chi)}_{k,S,T}(\psi^{-1},0)$.*

Remark 3.2. It is important to note that this conjecture is much weaker than the (relevant case of the) equivariant Tamagawa number conjecture. For example, if k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, then for any ψ that is trivial on the decomposition group in $\mathcal{G}_\chi$ of any p-adic place of k one has $r_{\psi,S}>r_\chi$ and so there is no interpolation condition at ψ specified above. When $r_\chi=0$, (the χ-component of) the element $\mathcal{L}_{K_\infty/k,S,T}$ is the p-adic L-function, and in the general case $r_\chi>0$, it plays a role of p-adic L-functions. We show later that Conjecture 3.1 can also be naturally interpreted in terms of the existence of suitable Iwasawa-theoretic measures (see Proposition 3.9).

3.2.2. In this subsection we assume that K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension and that K contains a primitive p-th root of unity and we briefly discuss how in this case the element $\mathcal{L}_{K_\infty/k,S,T}$ predicted by Conjecture IMC$(K_\infty/k,S,T,p)$ should encode information about the L-values at $s=n$ for arbitrary integers n.

To do this we use the twisting map

$$\mathrm{tw}:\Lambda\to\Lambda$$

defined by setting $\mathrm{tw}(\sigma):=\chi_{\mathrm{cyc}}(\sigma)\sigma$, where $\sigma\in\mathcal{G}$ and $\chi_{\mathrm{cyc}}:\mathcal{G}\to\mathbb{Z}_p^\times$ is the cyclotomic character. For an integer n, the ring Λ, which is regarded as a Λ-algebra via tw^n, is denoted by $\Lambda(n)$. For a finite extension L/k and a set of places Σ of k which contains $S_\infty(k)\cup S_{\mathrm{ram}}(L/k)\cup S_p(k)$, we put

$$C_{L,\Sigma}(n):=R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,\Sigma},\mathbb{Z}_p(n)),\mathbb{Z}_p)[-2].$$

For a set of places T of k which is disjoint from Σ, one can construct a canonical complex $C_{L,\Sigma,T}(n)$ which lies in the exact triangle

$$C_{L,\Sigma,T}(n)\to C_{L,\Sigma}(n)\to\bigoplus_{w\in T_L}H^1(\kappa(w),\mathbb{Z}_p(1-n))[0]$$

and is such that there exists a canonical isomorphism

$$\det_\Lambda(C_{K_\infty,S,T})\otimes_\Lambda\Lambda(n)\simeq\det_\Lambda(C_{K_\infty,S,T}(n)),$$

where $C_{K_\infty,S,T}(n)$ is defined by taking the inverse limit of the complexes $C_{K_m,S,T}(n)$ (see [16, Proposition 1.6.5(3)]).

Assuming the validity of IMC$(K_\infty/k,S,T,p)$, we then define $\mathcal{L}_{K_\infty/k,S,T}(n)$ to be the element of $\det_\Lambda(C_{K_\infty,S,T}(n))$ which corresponds to $\mathcal{L}_{K_\infty/k,S,T}\otimes 1$ under the above isomorphism and we denote the image of $\mathcal{L}_{K_\infty/k,S,T}(n)$ under the canonical surjection $\det_\Lambda(C_{K_\infty,S,T}(n))\to\det_{\mathbb{Z}_p[\Delta]}(C_{K,S,T}(n))$ by $\mathcal{L}_{K/k,S,T}(n)$.

Then the conjecture of Fukaya-Kato [16, Conjecture 2.3.2] suggests that $\mathcal{L}_{K/k,S,T}(n)$ is the zeta element for $(h^0(\mathrm{Spec}\,K)(n), \mathbb{Z}_p[\Delta])$, namely, the element which corresponds to the leading term

$$\theta^*_{K/k,S,T}(n) = \sum_{\chi\in\widehat{\Delta}} L^*_{k,S,T}(\chi^{-1}, n)e_\chi \in \mathbb{R}[\Delta]^\times \subset \mathbb{C}_p[\Delta]^\times$$

under the canonical isomorphism

$$\mathbb{C}_p\mathrm{det}_{\mathbb{Z}_p[\Delta]}(C_{K,S,T}(n)) \simeq \mathbb{C}_p \otimes_{\mathbb{Q}} \Xi(h^0(\mathrm{Spec}\,K)(n)) \simeq \mathbb{C}_p[\Delta],$$

where $\Xi(h^0(\mathrm{Spec}\,K)(n))$ is the fundamental line for $(h^0(\mathrm{Spec}\,K)(n), \mathbb{Q}[\Delta])$ (see [6, (29)]), and the first (resp. second) isomorphism is the p-adic regulator isomorphism $\tilde{\vartheta}_p$ in [7, p.479] (resp. the regulator isomorphism ϑ_∞ in [6, p.529]).

In a subsequent paper we shall study these subjects thoroughly. In particular, we generalize the Rubin-Stark conjecture to the setting of Tate motives $h^0(\mathrm{Spec}\,K)(n)$ for an arbitrary integer n (the original Rubin-Stark conjecture being regarded as the special case of this conjecture in the case $n = 0$).

We also show that the corresponding Rubin-Stark elements for twisted Tate motives are generalizations of Soulé's cyclotomic elements and we find that the above conjectural property of $\mathcal{L}_{K_\infty/k,S,T}$ predicts the existence of precise congruence relations between the Rubin-Stark elements for $h^0(\mathrm{Spec}\,K)(n)$ and $h^0(\mathrm{Spec}\,K)(n')$ for arbitrary integers n and n' which constitute a natural extension of the classical congruences of Kummer.

3.2.3. In the next result we record a useful invariance property of Conjecture 3.1. In the proof of this result we set

$$\delta_T := \prod_{v\in T}(1 - \mathrm{Fr}_v^{-1}\mathrm{N}v) \in \Lambda$$

where Fr_v denotes the arithmetic Frobenius in $\mathcal{G}$ of any place w of K_∞ that lies above v. This element belongs to $Q(\Lambda)^\times$ since for each $\chi \in \widehat{\Delta}$ and each $v \in T$ the image $1 - \mathrm{Fr}_v^{-1}\mathrm{N}v$ under the map $\Lambda \xrightarrow{\chi} \Lambda_\chi$ is non-zero.

Lemma 3.3. *The validity of Conjecture 3.1 is independent of the choice of T.*

Proof. It is enough to consider replacing T by a larger set T'. Set $T'' := T' \setminus T$. Then, one finds that there exists an exact triangle

$$C_{K_\infty,S,T} \to C_{K_\infty,S,T'} \to \bigoplus_{w\in T''_{K_\infty}} (\mathbb{Z}_p\kappa(w)^\times)[0]$$

($\kappa(w)^\times$ is defined by the inverse limit) and hence an equality

$$\mathrm{det}_\Lambda(C_{K_\infty,S,T'}) = \mathrm{det}_\Lambda(C_{K_\infty,S,T}) \prod_{v\in T''_{K_\infty}} \mathrm{Fitt}^0_\Lambda(\mathbb{Z}_p\kappa(w)^\times) = \mathrm{det}_\Lambda(C_{K_\infty,S,T})\delta_{T''},$$

where Fitt^0 denotes the (initial) Fitting ideal (see [25]).

Given this, it is straightforward to check that an element $\mathcal{L}_{K_\infty/k,S,T}$ validates Conjecture 3.1 with respect to T if and only if the element $\delta_{T''} \cdot \mathcal{L}_{K_\infty/k,S,T}$ validates Conjecture 3.1 with respect to T'. □

Following Lemma 3.3 we shall assume in the sequel that T contains two places of unequal residue characteristics and hence that each group $U_{L,S,T}$ is torsion-free.

3.2.4. For each Φ in $\bigwedge^{r_\chi} \mathrm{Hom}_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(U_{L_{\chi,n},S,T}, \mathbb{Z}_p[\mathcal{G}_{\chi,n}])$, Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ implies only that $\Phi(\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T})$ belongs to $\mathbb{Z}_p[\mathcal{G}_{\chi,n}]$.

By contrast, if Conjecture 3.1 is valid, then the following result shows that the elements $\Phi(\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T})$ encode significant arithmetic information.

In this result we write Fitt^a for the a-th Fitting ideal (see [25]).

Theorem 3.4. *Assume that the Iwasawa main conjecture (Conjecture 3.1) is valid for $(K_\infty/k, S, T)$. Then, for each $\chi \in \widehat{\Delta}$ and each positive integer n, we have*

$$\mathrm{Fitt}^{r_\chi}_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(H^1(C_{L_{\chi,n},S,T})) = \{\Phi(\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}) \mid \Phi \in \bigwedge^{r_\chi} \mathrm{Hom}_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(U_{L_{\chi,n},S,T}, \mathbb{Z}_p[\mathcal{G}_{\chi,n}])\}.$$

In particular, Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *is valid.*

Proof. The explicit definition of the elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$ implies directly that the assertion of Conjecture 3.1 is valid if and only if there is a Λ-basis $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ for which, for every character $\chi \in \widehat{\Delta}$ and every positive integer n, the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} e_{r_\chi}\mathbb{C}_p \bigwedge^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$.

Given this equivalence, the claimed result follows directly from Proposition 2.6(iii) and the same argument used to prove [9, Theorem 7.5]. □

3.2.5. For each character $\chi \in \widehat{\Delta}$, there is a natural ring homomorphism

$$\mathbb{Z}_p[[\mathcal{G}_\chi]] = \mathbb{Z}_p[[G_\chi \times \Gamma]] \xrightarrow{\chi} \mathbb{Z}_p[\mathrm{im}\,\chi][[\Gamma]] = \Lambda_\chi \subset Q(\Lambda_\chi).$$

In the sequel we use this homomorphism to regard $Q(\Lambda_\chi)$ as a $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-algebra.

In the next result we describe an important connection between the element $\mathcal{L}_{K_\infty/k,S,T}$ that is predicted to exist by Conjecture 3.1 and the inverse limit (over n) of the Rubin-Stark elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$. This result shows, in particular, that the element $\mathcal{L}_{K_\infty/k,S,T}$ in Conjecture 3.1 is unique (if it exists).

In the sequel we set

$$\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T} := \varprojlim_n \bigcap^{r_\chi} U_{L_{\chi,n},S,T},$$

where the inverse limit is taken with respect to the map

$$\bigcap^{r_\chi} U_{L_{\chi,m},S,T} \to \bigcap^{r_\chi} U_{L_{\chi,n},S,T}$$

induced by the norm map $U_{L_{\chi,m},S,T} \to U_{L_{\chi,n},S,T}$, where $n \le m$. Note that Rubin-Stark elements are norm compatible (see [27, Proposition 6.1] or [28, Proposition 3.5]), so if we know that Conjecture $\mathrm{RS}(L_{\chi,n}/k,S,T,V_\chi)_p$ is valid for all sufficiently large n, then we can define the element

$$\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T} := \varprojlim_n \epsilon^{V_\chi}_{L_{\chi,n}/k,S,T} \in \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}.$$

Theorem 3.5.

(i) *For each $\chi \in \widehat{\Delta}$, the homomorphism*

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \overset{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}}{\to} \bigcap^{r_\chi} U_{L_{\chi,n},S,T}$$

(see Proposition 2.6(iii)) induces an isomorphism of $Q(\Lambda_\chi)$-modules

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T} : \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq (\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

(ii) *If Conjecture 3.1 is valid, then we have*

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T}(\mathcal{L}_{K_\infty/k,S,T}) = \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}.$$

(Note that in this case Conjecture $\mathrm{RS}(L_{\chi,n}/k,S,T,V_\chi)_p$ *is valid for all n by Theorem 3.4.)*

Proof. Since the module $A^T_S(K_\infty) \otimes_\Lambda Q(\Lambda_\chi)$ vanishes, there are canonical isomorphisms

$$\begin{aligned} (1) \qquad & \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \\ \simeq\ & \det_{Q(\Lambda_\chi)}(C_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi)) \\ \simeq\ & \det_{Q(\Lambda_\chi)}(U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi)) \otimes_{Q(\Lambda_\chi)} \det^{-1}_{Q(\Lambda_\chi)}(\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_\chi)). \end{aligned}$$

It is also easy to check that there are natural isomorphisms

$$U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi) \simeq U_{L_{\chi,\infty},S,T} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi)$$

and

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_\chi) \simeq \mathcal{X}_{L_{\chi,\infty},S} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi) \simeq \mathcal{Y}_{L_{\chi,\infty},V_\chi} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi),$$

and that these are $Q(\Lambda_\chi)$-vector spaces of dimension $r := r_\chi (= \# V_\chi)$. The isomorphism (1) is therefore a canonical isomorphism of the form

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq (\bigwedge^r U_{L_{\chi,\infty},S,T} \otimes \bigwedge^r \mathcal{Y}^*_{L_{\chi,\infty},V_\chi}) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

Composing this isomorphism with the map induced by the non-canonical isomorphism

$$\bigwedge^r \mathcal{Y}^*_{L_\chi,\infty,V_\chi} \xrightarrow{\sim} \mathbb{Z}_p[[\mathcal{G}_\chi]]; w_1^* \wedge \cdots \wedge w_r^* \mapsto 1,$$

we have

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq (\bigwedge^r U_{L_\chi,\infty,S,T}) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

As in the proofs of Proposition 2.6(iii) and of [9, Lemma 4.3], this isomorphism is induced by $\varprojlim_n \pi^{V_\chi}_{L_{\chi,n}/k,S,T}$. Now the isomorphism in claim (i) is thus obtained directly from Lemma 3.6 below.

Claim (ii) follows by noting that the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} \bigcap^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$ (see the proof of Theorem 3.4). □

Lemma 3.6. *With notation as above, there is a canonical identification*

$$(\bigcap^r U_{L_\chi,\infty,S,T}) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi) = (\bigwedge^r U_{L_\chi,\infty,S,T}) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

Proof. Take a representative of $C_{L_\chi,\infty,S,T}$

$$\Pi_\infty \to \Pi_\infty$$

as in §2.4. Put $\Pi_n := \Pi_\infty \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} \mathbb{Z}_p[\mathcal{G}_{\chi,n}]$. We have

$$\bigcap^r U_{L_{\chi,n},S,T} = (\mathbb{Q}_p \bigwedge^r U_{L_{\chi,n},S,T}) \cap \bigwedge^r \Pi_n$$

(see Proposition 2.6(i)) and so $\varprojlim_n \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]} U_{L_{\chi,n},S,T}$ can be regarded as a submodule of the free $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-module

$$\varprojlim_n \bigwedge^r \Pi_n = \bigwedge^r \Pi_\infty.$$

For simplicity, we set

- $G_n := \mathcal{G}_{\chi,n}$,
- $G := \mathcal{G}_\chi$,
- $U_n := U_{L_{\chi,n},S,T}$,
- $U_\infty := U_{L_\chi,\infty,S,T}$,
- $Q := Q(\Lambda_\chi)$.

We show the equality

$$((\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n) \cap \bigwedge^r \Pi_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q = (\bigwedge^r U_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q$$

of the submodules of $(\bigwedge^r \Pi_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q$.

It is easy to see that

$$(\bigwedge^r U_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q \subset ((\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n) \cap \bigwedge^r \Pi_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q.$$

Conversely, take $a \in (\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n) \cap \bigwedge^r \Pi_\infty$ and set

$$M_n := \mathrm{coker}(U_n \to \Pi_n).$$

Then we have

$$\varprojlim_n M_n \simeq \mathrm{coker}(U_\infty \to \Pi_\infty) =: M_\infty.$$

Since

$$\Pi_\infty \otimes_{\mathbb{Z}_p[[G]]} Q \simeq (U_\infty \otimes_{\mathbb{Z}_p[[G]]} Q) \oplus (M_\infty \otimes_{\mathbb{Z}_p[[G]]} Q),$$

we have the decomposition

$$(\bigwedge^r \Pi_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q \simeq \bigoplus_{i=0}^r (\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q.$$

Write

$$a = (a_i)_i \in \bigoplus_{i=0}^r (\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q.$$

It is sufficient to show that $a_i = 0$ for all $i > 0$. We may assume that

$$a_i \in \mathrm{im}(\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty \to (\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q)$$

for every i. Since $a \in \bigwedge^r \Pi_\infty$, we can also write

$$a = (a_{(n)})_n \in \varprojlim_n \bigwedge^r \Pi_n.$$

For each n, we have a decomposition

$$\mathbb{Q}_p \bigwedge^r \Pi_n \simeq \bigoplus_{i=0}^r (\mathbb{Q}_p \bigwedge^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p \bigwedge^i M_n),$$

and we write

$$a_{(n)} = (a_{(n),i})_i \in \bigoplus_{i=0}^r (\mathbb{Q}_p \bigwedge^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p \bigwedge^i M_n).$$

Since $a \in \varprojlim_n \mathbb{Q}_p \bigwedge^r U_n$, we must have $a_{(n),i} = 0$ for all $i > 0$. To prove $a_i = 0$ for all $i > 0$, It is sufficient to show that the natural map

$$\text{(2)} \quad \operatorname{im}(\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty \to (\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q) \to \varprojlim_n (\mathbb{Q}_p \bigwedge^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p \bigwedge^{i} M_n)$$

is injective. Note that M_∞ is isomorphic to a submodule of Π_∞, since $M_\infty \simeq \ker(\Pi_\infty \to H^1(C_{L_{\chi,\infty},S,T}))$. Hence both U_∞ and M_∞ are embedded in Π_∞, and we have

$$\begin{aligned} &\ker(\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty \to (\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q) \\ &\qquad = \ker(\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty \xrightarrow{\alpha} (\bigwedge^{r} (\Pi_\infty \oplus \Pi_\infty)) \otimes_{\mathbb{Z}_p[[G]]} \Lambda_\chi). \end{aligned}$$

Set $\Lambda_{\chi,n} := \mathbb{Z}_p[\operatorname{im}\chi][\Gamma_{\chi,n}]$. The commutative diagram

$$\begin{tikzcd} \bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty \arrow[r, "\alpha"] \arrow[d, "\beta"] & (\bigwedge^{r} (\Pi_\infty \oplus \Pi_\infty)) \otimes_{\mathbb{Z}_p[[G]]} \Lambda_\chi \arrow[d, "f"] \\ \varprojlim_n \mathbb{Q}_p((\bigwedge^{r-i} U_n \otimes \bigwedge^{i} M_n) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}) \arrow[r, "g"'] & \varprojlim_n \mathbb{Q}_p((\bigwedge^{r} (\Pi_n \oplus \Pi_n)) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}) \end{tikzcd}$$

and the injectivity of f and g implies $\ker\alpha = \ker\beta$. Hence we have

$$\begin{aligned} & \ker(\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty \to (\bigwedge^{r-i} U_\infty \otimes \bigwedge^{i} M_\infty) \otimes_{\mathbb{Z}_p[[G]]} Q) \\ = & \ker\alpha \\ = & \ker\beta. \end{aligned}$$

This shows the injectivity of (2). $\square$

By Theorem 3.5, we can formulate the following conjecture, which is equivalent to Conjecture 3.1 under the assumption that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all $\chi \in \widehat{\Delta}$ and n.

Conjecture 3.7. *Assume that Conjecture* $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *is valid for all* $\chi \in \widehat{\Delta}$ *and* n*. Define* $\mathcal{L}_{K_\infty/k,S,T} \in \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda)$ *by*

$$\begin{aligned} \mathcal{L}_{K_\infty/k,S,T} &:= (\pi^{V_\chi,-1}_{L_{\chi,\infty}/k,S,T}(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}))_\chi \\ &\in \bigoplus_{\chi \in \widehat{\Delta}/\sim_{\mathbb{Q}_p}} (\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi)) \\ &= \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda). \end{aligned}$$

Then, we have

$$\Lambda \cdot \mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T}).$$

3.3. **Iwasawa main conjecture II.** In this section we reinterpret Conjecture 3.1 in terms of the existence of suitable Iwasawa-theoretic measures.

To do this we assume to be given, for each χ in $\widehat{\Delta}/\sim_{\mathbb{Q}_p}$, a homomorphism of $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-modules

$$\varphi_\chi : \bigwedge^{r_\chi} \mathcal{X}_{L_{\chi,\infty},S} \to \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}$$

for which $\ker(\varphi_\chi)$ is a torsion $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-module. The Rubin-Stark Conjecture implies the existence of a canonical such homomorphism φ_χ. Indeed, if we assume Conjecture $\mathrm{RS}(L_{\chi,n}/k,S,T,V_\chi)_p$ for all n, then we can define a homomorphism

$$\bigwedge^{r_\chi} \mathcal{X}_{L_{\chi,\infty},S} \to \bigwedge^{r_\chi} \mathcal{Y}_{L_{\chi,\infty},V_\chi} \to \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T},$$

where the first map is the natural surjection, and the second is given by

$$w_1 \wedge \cdots \wedge w_{r_\chi} \mapsto \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}.$$

Using Lemma 3.6, one sees that the kernel of this homomorphism is torsion, and that this homomorphism is canonical. For each character $\psi \in \widehat{\mathcal{G}_\chi}$ this homomorphism induces, upon taking coinvariance, a homomorphism of $\mathbb{Z}_p[G_\psi]$-modules

$$\varphi_{(\psi)} : \bigwedge^{r_\chi} \mathcal{X}_{L_\psi,S} \to \bigcap^{r_\chi} U_{L_\psi,S,T}.$$

Consider the endomorphism

$$e_\psi\mathbb{C}_p \bigwedge^{r_\chi} \mathcal{X}_{L_\psi,S} \stackrel{\varphi_{(\psi)}}{\to} e_\psi\mathbb{C}_p \bigwedge^{r_\chi} U_{L_\psi,S,T} \stackrel{\lambda^{-1}_{L_\psi,S}}{\to} e_\psi\mathbb{C}_p \bigwedge^{r_\chi} \mathcal{X}_{L_\psi,S}.$$

We denote the determinant of this endomorphism by $\mathcal{L}_\varphi(\psi)$.

In addition, since each Λ-module $\ker(\varphi_\chi)$ is torsion, the collection $\varphi = (\varphi_\chi)_\chi$ combines with the canonical isomorphisms (1) and the result of Lemma 3.6 to give a composite isomorphism of $Q(\Lambda)$-modules

$$\mu_\varphi : \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda) \simeq Q(\Lambda).$$

For each ψ in $\widehat{\mathcal{G}}$ we write $\mathfrak{q}_\psi$ for the kernel of the ring homomorphism $\psi : \Lambda \to \mathbb{C}_p$. This is a height one prime ideal of Λ and for any element λ of the localization $\Lambda_{\mathfrak{q}_\psi}$ there exists a non-zero divisor λ' of Λ for which both $\lambda'\lambda \in \Lambda$ and $\psi(\lambda') \neq 0$. In particular, in any such case the value of the quotient $\psi(\lambda'\lambda)/\psi(\lambda')$ is independent of the choice of λ' and will be denoted in the sequel by $\int \psi \, \mathrm{d}\lambda$.

Conjecture 3.8. *For any collection $\varphi = (\varphi_\chi)_\chi$ as above there exists a Λ-basis $\lambda_\varphi \in Q(\Lambda)$ of $\mu_\varphi(\det_\Lambda(C_{K^\infty,S,T}))$ such that for every $\chi \in \widehat{\Delta}/\sim_{\mathbb{Q}_p}$ and every $\psi \in \widehat{\mathcal{G}_\chi}$ for which both $r_{\psi,S} = r_\chi$ and $\mathcal{L}_\varphi(\psi) \neq 0$ one has $\lambda_\varphi \in \Lambda_{\mathfrak{q}_\psi}$ and*

$$\int \psi \mathrm{d}\lambda_\varphi = \mathcal{L}_\varphi(\psi) \cdot L_{k,S,T}^{(r_\chi)}(\psi^{-1}, 0). \tag{3}$$

Proposition 3.9. *Conjecture 3.1 is equivalent to Conjecture 3.8.*

Proof. Fix a Λ-basis $\mathcal{L}$ of $\det_\Lambda(C_{K^\infty,S,T})$. Then it is enough to prove that this element satisfies the interpolation conditions of Conjecture 3.1 if and only if for any choice of data φ as above, the element $\lambda_\varphi := \mu_\varphi(\mathcal{L})$ belongs to $\Lambda_{\mathfrak{q}_\psi}$ and satisfies the interpolation property (3).

Set $r := r_\chi$ and $V := V_\chi$. Then it is enough for us to fix a character $\psi \in \widehat{\mathcal{G}_\chi}$ for which $r_{\psi,S} = r$ and to show both that there exists a homomorphism φ_χ for which the map

$$e_\psi \mathbb{C}_p \bigwedge^r \mathcal{X}_{L_\psi,S} \stackrel{\varphi(\psi)}{\to} e_\psi \mathbb{C}_p \bigwedge^r U_{L_\psi,S,T}$$

is injective (and hence $\mathcal{L}_\varphi(\psi) \neq 0$) and also that for any such φ_χ there exists a commutative diagram of the form

$$\begin{array}{ccc} \det_\Lambda(C_{K^\infty,S,T}) & \xrightarrow{\mu_\varphi} & \mu_\varphi(\det_\Lambda(C_{K^\infty,S,T})) \\ \Big\downarrow{\scriptstyle \lambda_\psi} & & \Big\downarrow{\scriptstyle x \mapsto \int \psi \mathrm{d}x} \\ \mathbb{C}_p & \xrightarrow{\times \mathcal{L}_\varphi(\psi)} & \mathbb{C}_p. \end{array} \tag{4}$$

Set $\mathfrak{q} := \mathfrak{q}_\psi$. Then, since $\mathfrak{q}$ is a height one prime ideal of Λ which does not contain p the localization $\Lambda_\mathfrak{q}$ is a discrete valuation ring. In addition, ψ induces isomorphisms $\Lambda/\mathfrak{q} \simeq \mathbb{Z}_p[\mathrm{im}\,\psi]$ and $\Lambda_\mathfrak{q}/\mathfrak{q}\Lambda_\mathfrak{q} \simeq \mathbb{Q}_p(\psi) := \mathbb{Q}_p(\mathrm{im}\,\psi)$ and, if for each $\mathbb{Z}_p[G_\psi]$-module M we set $M_\psi := e_\psi(\mathbb{Q}_p(\psi) \otimes_{\mathbb{Z}_p} M)$, then we have an isomorphism of $\mathbb{Q}_p(\psi)$-vector spaces

$$\begin{aligned} H^1(C_{L_{\chi,\infty},S,T})_\mathfrak{q}/\mathfrak{q}\Lambda_\mathfrak{q} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C_{L_{\chi,\infty},S,T})/\mathfrak{q} &\simeq H^1(C_{L_\psi,S,T})_\psi \\ &= (\mathcal{X}_{L_\psi,S})_\psi = (\mathcal{Y}_{L_\psi,V})_\psi. \end{aligned} \tag{5}$$

Here the second equality follows from the exact sequence

$$0 \to A_S^T(L_{\chi,\infty}) \to H^1(C_{L_{\chi,\infty},S,T}) \stackrel{\pi}{\to} \mathcal{X}_{L_{\chi,\infty},S} \to 0 \tag{6}$$

and the third from the assumption $r_{\psi,S} = r$.

Since $\mathcal{Y}_{L_{\chi,\infty},V,\mathfrak{q}}$ is a free $\Lambda_\mathfrak{q}$-module there is a direct sum decomposition $\mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}} = \mathcal{X}_{L_{\chi,\infty},S\setminus V,\mathfrak{q}} \oplus \mathcal{Y}_{L_{\chi,\infty},V,\mathfrak{q}}$. Thus, since the composite isomorphism (5) factors through the map π in (6), the module $\mathcal{X}_{L_{\chi,\infty},S\setminus V,\mathfrak{q}}/\mathfrak{q}\Lambda_\mathfrak{q}$ vanishes, and hence also (by Nakayama's Lemma) the module $\mathcal{X}_{L_{\chi,\infty},S\setminus V,\mathfrak{q}}$ vanishes.

The $\Lambda_{\mathfrak{q}}$-module $\mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}} = \mathcal{Y}_{L_{\chi,\infty},V,\mathfrak{q}}$ is therefore free of rank r and, given this, the $\mathfrak{q}$-localisation of the tautological exact sequence

$$0 \to U_{L_{\chi,\infty},S,T} \to \Pi_\infty \to \Pi_\infty \to H^1(C_{L_{\chi,\infty},S,T}) \to 0$$

implies $U_{L_{\chi,\infty},S,T,\mathfrak{q}}$ is also a free $\Lambda_{\mathfrak{q}}$-module of rank r. In particular, since the $\Lambda_{\mathfrak{q}}$-modules $\mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}}$ and $U_{L_{\chi,\infty},S,T,\mathfrak{q}}$ are isomorphic we may choose a homomorphism of Λ-modules $\varphi'_\chi : \mathcal{X}_{L_{\chi,\infty},S} \to U_{L_{\chi,\infty},S,T}$ with the property that $\ker(\varphi'_\chi)_{\mathfrak{q}}$ vanishes. It is then easily checked that the r-th exterior power of φ'_χ induces a homomorphism of the required sort φ_χ for which the induced map

$$e_\psi \mathbb{C}_p \bigwedge^r \mathcal{X}_{L_\psi,S} \stackrel{\varphi(\psi)}{\to} e_\psi \mathbb{C}_p \bigwedge^r U_{L_\psi,S,T}$$

is injective.

To prove the existence of a commutative diagram (4) we note first that, since the $\Lambda_{\mathfrak{q}}$-module $\mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}} = \mathcal{Y}_{L_{\chi,\infty},V,\mathfrak{q}}$ is free, the exact sequence (6) splits and so the isomorphism (5) combines with Nakayama's Lemma to imply $A_S^T(L_{\chi,\infty})_{\mathfrak{q}}$ vanishes. It follows that $H^0(C_{L_{\chi,\infty},S,T})_{\mathfrak{q}} = U_{L_{\chi,\infty},S,T,\mathfrak{q}}$ and $H^1(C_{L_{\chi,\infty},S,T})_{\mathfrak{q}} = \mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}}$ are both free $\Lambda_{\mathfrak{q}}$-modules of rank r. This gives a canonical isomorphism of $\Lambda_{\mathfrak{q}}$-modules

$$\mathrm{det}_\Lambda(C_{K^\infty,S,T})_{\mathfrak{q}} \simeq (\bigwedge^r_{\Lambda_{\mathfrak{q}}} U_{L_{\chi,\infty},S,T,\mathfrak{q}}) \otimes_{\Lambda_{\mathfrak{q}}} (\bigwedge^r_{\Lambda_{\mathfrak{q}}} \mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}})^*.$$

and by combining this isomorphism with the natural projection map

$$(\bigwedge^r_{\Lambda_{\mathfrak{q}}} U_{L_{\chi,\infty},S,T,\mathfrak{q}}) \otimes_{\Lambda_{\mathfrak{q}}} (\bigwedge^r_{\Lambda_{\mathfrak{q}}} \mathcal{X}_{L_{\chi,\infty},S,\mathfrak{q}})^* \to (\bigwedge^r_{\mathbb{Q}_p(\psi)} (U_{L_\psi,S,T})_\psi) \otimes_{\mathbb{Q}_p(\psi)} (\bigwedge^r_{\mathbb{Q}_p(\psi)} (\mathcal{X}_{L_\psi,S})_\psi)^*$$

we obtain the horizontal arrow in the following diagram.

$$\begin{tikzcd}
 & & \mathbb{C}_p \\
\mathrm{det}_\Lambda(C_{K^\infty,S,T}) \arrow[r] \arrow[urr, "(x\mapsto\int\psi\mathrm{d}x)\circ\mu_\varphi"] \arrow[drr, "\lambda_\psi"'] & (\bigwedge^r_{\mathbb{Q}_p(\psi)}(U_{L_\psi,S,T})_\psi) \otimes_{\mathbb{Q}_p(\psi)} (\bigwedge^r_{\mathbb{Q}_p(\psi)}(\mathcal{X}_{L_\psi,S})_\psi)^* \arrow[ur, "\theta_1"'] \arrow[dr, "\theta_2"] & \\
 & & \mathbb{C}_p \arrow[uu, "\times\mathcal{L}_\varphi(\psi)"']
\end{tikzcd}$$

Here θ_1 and θ_2 are the maps induced by $\varphi_{(\psi)}^{-1}$ and $\lambda_{L_\psi,S}$ and the respective evaluation maps. The commutativity of the right hand triangle is then clear and the commutativity of the two remaining triangles follows by an explicit comparison of the definitions of the maps involved. Since the whole diagram commutes this then gives a commutative diagram of the form (4), as required. □

3.4. **Iwasawa main conjecture III.** In this subsection, we work under the following simplifying assumptions:

$$(*)\begin{cases} p \text{ is odd,} \\ \text{for every } \chi \in \widehat{\Delta},\ V_\chi \text{ contains no finite places.} \end{cases}$$

We note that the second assumption here is satisfied whenever k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension.

3.4.1. We start by quickly reviewing some basic facts concerning the height one prime ideals of Λ.

We say that a height one prime ideal $\mathfrak{p}$ of Λ is 'regular' (resp. 'singular') if one has $p \notin \mathfrak{p}$ (resp. $p \in \mathfrak{p}$). We will often abbreviate 'height one regular (resp. singular) prime ideal' to 'regular (resp. singular) prime'.

If $\mathfrak{p}$ is regular, then $\Lambda_\mathfrak{p}$ is identified with the localization of $\Lambda[1/p]$ at $\mathfrak{p}\Lambda[1/p]$. Since we have the decomposition

$$\Lambda\left[\frac{1}{p}\right] = \bigoplus_{\chi\in\widehat{\Delta}/\sim_{\mathbb{Q}_p}} \Lambda_\chi\left[\frac{1}{p}\right], \tag{7}$$

we see that $\Lambda_\mathfrak{p}$ is equal to the localization of some $\Lambda_\chi[1/p]$ at $\mathfrak{p}\Lambda_\chi[1/p]$. This shows that $Q(\Lambda_\mathfrak{p}) = Q(\Lambda_\chi)$. This $\chi \in \widehat{\Delta}/\sim_{\mathbb{Q}_p}$ is uniquely determined by $\mathfrak{p}$, so we denote it by $\chi_\mathfrak{p}$. Since $\Lambda_\chi[1/p]$ is a regular local ring, we also see that $\Lambda_\mathfrak{p}$ is a one-dimensional regular local ring i.e. discrete valuation ring.

Next, suppose that $\mathfrak{p}$ is a singular prime. We have the decomposition

$$\Lambda = \bigoplus_{\chi\in\widehat{\Delta}'/\sim_{\mathbb{Q}_p}} \mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]],$$

where Δ_p is the Sylow p-subgroup of Δ, and Δ' is the unique subgroup of Δ which is isomorphic to Δ/Δ_p. From this, we see that $\Lambda_\mathfrak{p}$ is identified with the localization of some $\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]]$ at $\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]]$. By [8, Lemma 6.2(i)], we have

$$\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]] = (\sqrt{p\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p]}),$$

where we denote the radical of an ideal I by $\sqrt{I}$. This shows that there is a one-to-one correspondence between the set of all singular primes of Λ and the set $\widehat{\Delta}'/\sim_{\mathbb{Q}_p}$. We

denote by $\chi_{\mathfrak{p}} \in \widehat{\Delta}'/{\sim_{\mathbb{Q}_p}}$ the character corresponding to $\mathfrak{p}$. The next lemma shows that

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi \in \widehat{\Delta}/\sim_{\mathbb{Q}_p}, \chi|_{\Delta'} = \chi_{\mathfrak{p}}} Q(\Lambda_{\chi}).$$

Lemma 3.10. *Let $E/\mathbb{Q}_p$ be a finite unramified extension, and $\mathcal{O}$ be its ring of integers. Let P be a finite abelian group whose order is a power of p. Put $\Lambda := \mathcal{O}[P][[\Gamma]]$ and $\mathfrak{p} := \sqrt{p\mathcal{O}[P]}\Lambda$. ($\mathfrak{p}$ is the unique singular prime of Λ.) Then we have*

$$Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda) = \bigoplus_{\chi \in \widehat{P}/\sim_E} Q(\mathcal{O}[\operatorname{im}\chi][[\Gamma]]).$$

Proof. Note that p is not a zero divisor of Λ, so we have

$$Q(\Lambda_{\mathfrak{p}}) = Q\left(\Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right]\right).$$

We have the decomposition

$$\Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right] = \bigoplus_{\chi \in \widehat{P}/\sim_E} e_{\chi}\Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right],$$

where $e_{\chi} := \sum_{\chi' \sim_E \chi} e_{\chi'}$. It is easy to see that each $e_{\chi}\Lambda_{\mathfrak{p}}[1/p]$ is a domain. Therefore we have

$$Q\left(\Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right]\right) = \bigoplus_{\chi \in \widehat{P}/\sim_E} Q\left(e_{\chi}\Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right]\right).$$

For $\chi \in \widehat{P}/{\sim_E}$, put $\mathfrak{q}_{\chi} := \ker(\Lambda \xrightarrow{\chi} \mathcal{O}[\operatorname{im}\chi][[\Gamma]])$. Note that $\sqrt{p\mathcal{O}[P]} = (p, I_{\mathcal{O}}(P))$, where $I_{\mathcal{O}}(P)$ is the kernel of the augmentation map $\mathcal{O}[P] \to \mathcal{O}$. This can be shown as follows. Note that any prime ideal of $\mathcal{O}/p\mathcal{O}[P]$ is the kernel of some surjection $f : \mathcal{O}/p\mathcal{O}[P] \to R$ with some finite domain R. It is well-known that every finite domain is a field, so we must have $R \simeq \mathcal{O}/p\mathcal{O}$, and f is the augmentation map $\mathcal{O}/p\mathcal{O}[P] \to \mathcal{O}/p\mathcal{O} \simeq R$. This shows that $\ker f$ is the unique prime ideal of $\mathcal{O}/p\mathcal{O}[P]$. Hence we have $\sqrt{p\mathcal{O}[P]} = (p, I_{\mathcal{O}}(P))$. From this, we also see that

$$\sqrt{p\mathcal{O}[P]} = \ker(\mathcal{O}[P] \xrightarrow{\chi} \mathcal{O}[\operatorname{im}\chi] \to \mathcal{O}[\operatorname{im}\chi]/\pi_{\chi}\mathcal{O}[\operatorname{im}\chi] \simeq \mathcal{O}/p\mathcal{O})$$

holds for any $\chi \in \widehat{P}/{\sim_E}$, where $\pi_{\chi} \in \mathcal{O}[\operatorname{im}\chi]$ is a uniformizer. This shows that $\mathfrak{q}_{\chi} \subset \mathfrak{p}$. Hence, we know that $\Lambda_{\mathfrak{q}_{\chi}}$ is the localization of $\Lambda_{\mathfrak{p}}[1/p]$ at $\mathfrak{q}_{\chi}\Lambda_{\mathfrak{p}}[1/p]$. One can check that $\Lambda_{\mathfrak{q}_{\chi}} = Q(e_{\chi}\Lambda_{\mathfrak{p}}[1/p])$. Since we have $\Lambda_{\mathfrak{q}_{\chi}} = Q(\mathcal{O}[\operatorname{im}\chi][[\Gamma]])$, the lemma follows. □

For a height one prime ideal $\mathfrak{p}$ of Λ, define a subset $\Upsilon_{\mathfrak{p}} \subset \widehat{\Delta}/{\sim_{\mathbb{Q}_p}}$ by

$$\Upsilon_{\mathfrak{p}} := \begin{cases} \{\chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is regular,} \\ \{\chi \in \widehat{\Delta}/\sim_{\mathbb{Q}_p} \mid \chi|_{\Delta'} = \chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is singular.} \end{cases}$$

The above argument shows that

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} Q(\Lambda_\chi).$$

To end this section we recall a useful result concerning μ-invariants.

Lemma 3.11. *Let M be a finitely generated torsion Λ-module. Let $\mathfrak{p}$ be a singular prime of Λ. Then the following are equivalent:*

(i) *The μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_{\mathfrak{p}}} M$ vanishes.*
(ii) *For any $\chi \in \Upsilon_{\mathfrak{p}}$, the μ-invariant of the $\mathbb{Z}_p[\mathrm{im}\,\chi][[\Gamma]]$-module $M \otimes_{\mathbb{Z}_p[\Delta']} \mathbb{Z}_p[\mathrm{im}\,\chi]$ vanishes.*
(iii) *$M_{\mathfrak{p}} = 0$.*

Proof. See [14, Lemma 5.6]. □

3.4.2. In the rest of this subsection we assume the condition $(*)$.

Lemma 3.12. *Let $\mathfrak{p}$ be a singular prime of Λ. Then V_χ is independent of $\chi \in \Upsilon_{\mathfrak{p}}$. In particular, for any $\chi \in \Upsilon_{\mathfrak{p}}$, the $Q(\Lambda_{\mathfrak{p}})$-module $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$ is free of rank r_χ.*

Proof. It is sufficient to show that $V_\chi = V_{\chi_{\mathfrak{p}}}$ for any $\chi \in \Upsilon_{\mathfrak{p}}$. Note that the extension degree $[L_{\chi,\infty} : L_{\chi_{\mathfrak{p}},\infty}] = [L_\chi : L_{\chi_{\mathfrak{p}}}]$ is a power of p. Since p is odd by the assumption $(*)$, we see that an infinite place of k which splits completely in $L_{\chi_{\mathfrak{p}},\infty}$ also splits completely in $L_{\chi,\infty}$. By the assumption $(*)$, we know every places in $V_{\chi_{\mathfrak{p}}}$ is infinite. Hence we have $V_\chi = V_{\chi_{\mathfrak{p}}}$. □

The above result motivates us, for any height one prime ideal $\mathfrak{p}$ of Λ, to define $V_{\mathfrak{p}} := V_\chi$ and $r_{\mathfrak{p}} := r_\chi$ by choosing some $\chi \in \Upsilon_{\mathfrak{p}}$.

Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \widehat{\Delta}$ and n. We then define the '$\mathfrak{p}$-part' of the Rubin-Stark element

$$\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \in \Big(\bigwedge^{r_{\mathfrak{p}}} U_{K_\infty,S,T}\Big) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$$

as the image of

$$(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T})_{\chi \in \Upsilon_{\mathfrak{p}}} \in \bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} \bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T}$$

under the natural map

$$\bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} \bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T} \to \bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} \Big(\bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi) = \Big(\bigwedge^{r_{\mathfrak{p}}} U_{K_\infty,S,T}\Big) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}}).$$

(see Lemma 3.6.)

We can now formulate a much more explicit main conjecture.

Conjecture 3.13. *If condition (∗) is valid, then for every height one prime ideal $\mathfrak{p}$ of Λ there is an equality*

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^0_\Lambda(A^T_S(K_\infty))\mathrm{Fitt}^0_\Lambda(\mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}}) \cdot (\bigwedge\nolimits^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T})_{\mathfrak{p}}.$$

Remark 3.14. At every height one prime ideal $\mathfrak{p}$ there is an equality

$$\mathrm{Fitt}^0_\Lambda(A^T_S(K_\infty))_{\mathfrak{p}}\mathrm{Fitt}^0_\Lambda(\mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}})_{\mathfrak{p}} = \mathrm{Fitt}^{r_{\mathfrak{p}}}_\Lambda(H^1(C_{K_\infty,S,T}))_{\mathfrak{p}}.$$

If $\mathfrak{p}$ is regular, then $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring and this equality follows directly from the exact sequence

$$0 \to A^T_S(K_\infty) \to H^1(C_{K_\infty,S,T}) \to \mathcal{X}_{K_\infty,S} \to 0.$$

If $\mathfrak{p}$ is singular, then the equality is valid since the result of Lemma 3.11 implies $(\mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}})_{\mathfrak{p}}$ vanishes and so $H^1(C_{K_\infty,S,T})_{\mathfrak{p}}$ is isomorphic to the direct sum $A^T_S(K_\infty)_{\mathfrak{p}} \oplus (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$.

Conjecture 3.13 is thus valid if and only if for every height one prime $\mathfrak{p}$ one has

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^{r_{\mathfrak{p}}}_\Lambda(H^1(C_{K_\infty,S,T})) \cdot (\bigwedge\nolimits^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T})_{\mathfrak{p}}.$$

Remark 3.15. If the prime $\mathfrak{p}$ is singular, then $(\mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}})_{\mathfrak{p}}$ vanishes and one has $\mathrm{Fitt}^0_\Lambda(A^T_S(K_\infty))_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$ if and only if the μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_{\mathfrak{p}}}A^T_S(K_\infty)$ vanishes (see Lemma 3.11). Thus, for any such $\mathfrak{p}$ Conjecture 3.13 implies that the invariant of $e_{\chi_{\mathfrak{p}}}A^T_S(K_\infty)$ vanishes if and only if one has

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = (\bigwedge\nolimits^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T})_{\mathfrak{p}}. \tag{8}$$

In a similar way, one finds that the vanishing of the μ-invariant of $e_{\chi_{\mathfrak{p}}}A^T_S(K_\infty)$ implies that Conjecture 3.13 for $\mathfrak{p}$ is itself equivalent to the equality (8).

Remark 3.16. For every height one prime ideal $\mathfrak{p}$ of Λ, put $\epsilon^{\mathfrak{p}}_{K_\infty/k,S} := \delta_T^{-1} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}$. Then Lemma 3.3 implies that the equality of Conjecture 3.13 is valid at $\mathfrak{p}$ if and only if one has

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S} = \mathrm{Fitt}^0_\Lambda(A_S(K_\infty))\,\mathrm{Fitt}^0_\Lambda(\mathcal{X}_{K_\infty,S\setminus S_\infty}) \cdot (\bigwedge\nolimits^{r}_\Lambda U_{K_\infty,S})_{\mathfrak{p}}.$$

3.4.3. Before comparing Conjecture 3.13 to the more general Conjecture 3.1 we show that the assumed validity of the p-part of the Rubin-Stark conjecture already gives strong evidence in favour of Conjecture 3.13.

We note, in particular, that if $\mathfrak{p}$ is a singular prime of Λ (and an appropriate μ-invariant vanishes), then the inclusion proved in the following result constitutes 'one half' of the equality (8) that is equivalent in this case to Conjecture 3.13.

Proposition 3.17. *Let $\mathfrak{p}$ be a height one prime ideal of Λ. When $\mathfrak{p}$ is singular, assume that the μ-invariant of $e_{\chi_{\mathfrak{p}}}A^T_S(K_\infty)$ (as $\mathbb{Z}_p[[\Gamma]]$-module) vanishes. Then the following claims are valid.*

(i) *The $\Lambda_{\mathfrak{p}}$-module $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is free of rank $r_{\mathfrak{p}}$.*

(ii) *If Conjecture* $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *is valid for every* χ *in* $\widehat{\Delta}$ *and every natural number* n*, then there is an inclusion*

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \subset (\bigwedge\nolimits_\Lambda^{r_{\mathfrak{p}}} U_{K_\infty,S,T})_{\mathfrak{p}}.$$

Proof. As in the proof of Lemma 3.6, we choose a representative of $C_{K_\infty,S,T}$

$$\Pi_\infty \stackrel{\psi_\infty}{\to} \Pi_\infty.$$

We have the exact sequence

$$0 \to U_{K_\infty,S,T} \to \Pi_\infty \stackrel{\psi_\infty}{\to} \Pi_\infty \to H^1(C_{K_\infty,S,T}) \to 0. \tag{9}$$

If $\mathfrak{p}$ is regular, then $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring and the exact sequence (9) implies that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $\mathrm{im}(\psi_\infty)_{\mathfrak{p}}$ are free. Since $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$ is isomorphic to $\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$, we also know that the rank of $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is $r_{\mathfrak{p}}$.

Suppose next that $\mathfrak{p}$ is singular. Since the μ-invariant of $e_{\chi_{\mathfrak{p}}}\mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}}$ vanishes, we apply Lemma 3.11 to deduce that $(\mathcal{X}_{K_\infty,S})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$. In a similar way, the assumption that the μ-invariant of $e_{\chi_{\mathfrak{p}}}A_S^T(K_\infty)$ vanishes implies that $A_S^T(K_\infty)_{\mathfrak{p}} = 0$. Hence we have $H^1(C_{K_\infty,S,T})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$. By assumption $(*)$, we know that $\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}}$ is projective as Λ-module. This implies that $H^1(C_{K_\infty,S,T})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$ is a free $\Lambda_{\mathfrak{p}}$-module of rank $r_{\mathfrak{p}}$. By choosing splittings of the sequence (9), we then easily deduce that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $\mathrm{im}(\psi_\infty)_{\mathfrak{p}}$ are free and that the rank of $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is equal to $r_{\mathfrak{p}}$.

At this stage we have proved that, for any height one prime ideal $\mathfrak{p}$ of Λ, the $\Lambda_{\mathfrak{p}}$-module $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is both free of rank $r_{\mathfrak{p}}$ (as required to prove claim (i)) and also a direct summand of $(\Pi_\infty)_{\mathfrak{p}}$, and hence that

$$(\bigwedge\nolimits_\Lambda^{r_{\mathfrak{p}}} U_{K_\infty,S,T})_{\mathfrak{p}} = (\bigwedge\nolimits_\Lambda^{r_{\mathfrak{p}}} U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})) \cap (\bigwedge\nolimits_\Lambda^{r_{\mathfrak{p}}} \Pi_\infty)_{\mathfrak{p}}. \tag{10}$$

Now we make the stated assumption concerning the validity of the p-part of the Rubin-Stark conjecture. This implies, by the proof of Theorem 3.5(i), that for each $\mathfrak{p}$ the element $\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}$ lies in both $(\bigwedge_\Lambda^{r_{\mathfrak{p}}} \Pi_\infty)_{\mathfrak{p}}$ and

$$\bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} (\bigwedge\nolimits_\Lambda^{r_\chi} U_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) = (\bigwedge\nolimits_\Lambda^{r_{\mathfrak{p}}} U_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}}),$$

and hence, by (10) that it belongs to $(\bigwedge_\Lambda^{r_{\mathfrak{p}}} U_{K_\infty,S,T})_{\mathfrak{p}}$, as required to prove claim (ii). $\square$

In the next result we compare Conjecture 3.13 to the more general Conjecture 3.1.

Proposition 3.18. *Assume that Conjecture* $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *holds for all characters* χ *in* $\widehat{\Delta}$ *and all sufficiently large* n *and that for each character* χ *in* $\widehat{\Delta}'/\sim_{\mathbb{Q}_p}$ *the* μ*-invariant of the* $\mathbb{Z}_p[[\Gamma]]$*-module* $e_\chi A_S^T(K_\infty)$ *vanishes. Then Conjectures 3.1 and 3.13 are equivalent.*

Proof. Since $\det_\Lambda(C_{K_\infty,S,T})$ is an invertible Λ-module the equality $\Lambda\cdot\mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})$ in Conjecture 3.1 is valid if and only if at every height one prime ideal $\mathfrak{p}$ of Λ one has

$$\Lambda_\mathfrak{p}\cdot\mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})_\mathfrak{p} \tag{11}$$

(see [8, Lemma 6.1]).

If $\mathfrak{p}$ is regular, then one easily sees that this equality is valid if and only if the equality

$$\Lambda_\mathfrak{p}\cdot\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^{r_\mathfrak{p}}_\Lambda(H^1(C_{K_\infty,S,T}))\cdot\Big({\bigwedge}^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\Big)_\mathfrak{p}$$

is valid, by using Theorem 3.5(ii).

If $\mathfrak{p}$ is singular, then the assumption of vanishing μ-invariants and the argument in the proof of Proposition 3.17(i) shows that the $\Lambda_\mathfrak{p}$-modules $(U_{K_\infty,S,T})_\mathfrak{p}$ and $H^1(C_{K_\infty,S,T})_\mathfrak{p}$ are both free of rank $r_\mathfrak{p}$. Noting this, we see that (11) holds if and only if one has

$$\Lambda_\mathfrak{p}\cdot\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \Big({\bigwedge}^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\Big)_\mathfrak{p}$$

and so in this case the claimed result follows from Remark 3.15. □

3.4.4. In our earlier paper [9] we defined canonical Selmer modules $\mathcal{S}_{S,T}(\mathbb{G}_{m/F})$ and $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F})$ for $\mathbb{G}_m$ over number fields F that are of finite degree over $\mathbb{Q}$. For any intermediate field L of K_∞/k, we now set

$$\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_F \mathcal{S}_{S,T}(\mathbb{G}_{m/F})\otimes\mathbb{Z}_p,\quad \mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_F \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F})\otimes\mathbb{Z}_p$$

where in both limits F runs over all finite extensions of k in L and the transition morphisms are the natural corestriction maps.

We note in particular that, by its very definition, $\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})$ coincides with $H^1(C_{L,S,T})$. In addition, this definition implies that for any subset V of S comprising places that split completely in L the kernel of the natural (composite) projection map

$$\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V := \ker(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})\to\mathcal{X}_{L,S}\to\mathcal{Y}_{L,V})$$

lies in a canonical exact sequence of the form

$$0\to A^T_S(L)\to\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V\to\mathcal{X}_{L,S\setminus V}\to 0. \tag{12}$$

We now interpret our Iwasawa main conjecture in terms of characteristic ideals.

Conjecture 3.19. *Assume Conjecture* $\mathrm{RS}(L_{\chi,n}/k,S,T,V_\chi)_p$ *holds for all* $\chi\in\widehat{\Delta}$ *and all non-negative integers* n. *Then for any* $\chi\in\widehat{\Delta}$ *there are equalities*

$$\begin{aligned}\mathrm{char}_{\Lambda_\chi}\Big(\Big({\bigcap}^{r_\chi} U_{L_{\chi,\infty},S,T}/\langle\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}\rangle\Big)^\chi\Big) &= \mathrm{char}_{\Lambda_\chi}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L_{\chi,\infty}})^\chi_{V_\chi})\\ &= \mathrm{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^\chi)\mathrm{char}_{\Lambda_\chi}((\mathcal{X}_{L_{\chi,\infty},S\setminus V_\chi})^\chi).\end{aligned} \tag{13}$$

Here, for any $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-module M we write M^χ for the Λ_χ-module $M \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Z}_p[\mathrm{im}\,\chi]$ and $\mathrm{char}_{\Lambda_\chi}(M^\chi)$ for its characteristic ideal in Λ_χ. In addition, the second displayed equality is a direct consequence of the appropriate case of the exact sequence (12).

Proposition 3.20. *Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all characters χ in $\widehat{\Delta}$ and all sufficiently large natural numbers n and that for each character $\chi \in \widehat{\Delta}'/\sim_{\mathbb{Q}_p}$ the μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_\chi A_S^T(K_\infty)$ vanishes. Then Conjectures 3.1 is equivalent to Conjecture 3.19.*

Proof. Note that by our assumption $\mu = 0$ we have $(\bigcap^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p} = (\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}$ for any height one prime $\mathfrak{p}$, using (10). Therefore, Conjecture 3.13 implies the equality (13) for any χ.

On the other hand, for a height one regular prime $\mathfrak{p}$, we can regard $\mathfrak{p}$ to be a prime of Λ_χ for some χ, so the equality (13) implies the equality in Conjecture 3.13. For a singular prime $\mathfrak{p}$, by Lemma 3.11, (13) for any χ implies $(\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}/\langle \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}\rangle = 0$, thus Conjecture 3.13.

The proposition therefore follows from Proposition 3.18. □

3.5. **The case of CM-fields.** In this section, we use the following strengthening of the condition $(*)$ used above.

$$(**) \begin{cases} p \text{ is odd,} \\ k \text{ is totally real and } K \text{ is either totally real or a CM-field,} \\ k_\infty/k \text{ is the cyclotomic } \mathbb{Z}_p\text{-extension.} \end{cases}$$

Under this hypothesis Iwasawa has conjectured that for every $\chi \in \widehat{\Delta}'/\sim_{\mathbb{Q}_p}$ the μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_\chi A_S^T(K_\infty)$ vanishes and, if this is true, then Proposition 3.18 implies that the Conjectures 3.1 and 3.13 are equivalent.

In addition, in this case we can use the main results of Wiles [36] and of Büyükboduk in [10] to give the following concrete evidence in support of these conjectures.

In the following we denote $S_\infty(k)$ and $S_p(k)$ simply by S_∞ and S_p respectively.

Theorem 3.21. *Assume the condition (∗∗).*

(i) *If K is a CM-field and the μ-invariant of K_∞/K vanishes, then the minus part of Conjecture 3.1 is valid for $(K_\infty/k, S, T)$.*

(ii) *Suppose that χ is an even character. Then the equality of Conjecture 3.19 is valid for χ whenever all of the following conditions are satisfied:*
   (a) *all $v \in S_p$ are unramified in L_χ,*
   (b) *$k/\mathbb{Q}$ is unramified at p,*
   (c) *every $v \in S \setminus S_\infty$ satisfies $\chi(G_v) \neq 1$,*
   (d) *the order of χ is prime to p,*
   (e) *with T chosen as in* [10, Remark 3.1]*, the Rubin-Stark conjecture holds for $(F/k, S, T, S_\infty)$ for all F in the set $\mathcal{K}$ of finite abelian extensions of k that is defined in* [10, §3] *(where our L_χ corresponds to the field L),*

(f) *the Leopoldt conjecture holds for $L_{\chi,n}$ for all positive integer n.*

3.5.1. We obtain Theorem 3.21(i) as a straightforward consequence of the main conjecture proved by Wiles [36]. In fact, for an odd character χ, one has $r_\chi = 0$ and the Rubin-Stark elements are Stickelberger elements. Therefore, $\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}$ is the p-adic L-function of Deligne-Ribet.

We shall prove the equality (13) in Conjecture 3.19 for each odd $\chi \in \widehat{\Delta}$. We fix such a character χ, and may take $K = L_\chi$ and $S = S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$. Let S'_p be the set of p-adic primes which split completely in K. If $v \in S \setminus V_\chi$ is prime to p, it is ramified in $L_\chi = K$, so we have $\mathrm{char}_{\Lambda_\chi}(\mathcal{X}^\chi_{L_{\chi,\infty},S\setminus V_\chi}) = \mathrm{char}_{\Lambda_\chi}(\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p})$. Let $A^T(L_{\chi,\infty})$ be the inverse limit of the p-component of the T-ray class group of the full integer ring of $L_{\chi,n}$. By sending the prime w above v in S'_p to the class of w, we obtain a homomorphism $\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p} \longrightarrow A^T(L_{\chi,\infty})^\chi$, which is known to be injective. Since the sequence

$$\mathcal{Y}^\chi_{L_{\chi,\infty},S} \longrightarrow A^T(L_{\chi,\infty})^\chi \longrightarrow A^T_S(L_{\chi,\infty})^\chi \longrightarrow 0$$

is exact and the kernel of $\mathcal{Y}^\chi_{L_{\chi,\infty},S} \longrightarrow \mathcal{Y}^\chi_{L_{\chi,\infty},S'_p}$ is finite, we have

$$\mathrm{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^\chi)\mathrm{char}_{\Lambda_\chi}((\mathcal{Y}_{L_{\chi,\infty},S})^\chi) = \mathrm{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi).$$

Therefore, by noting $\chi \neq 1$, the equality (13) in Conjecture 3.19 becomes

$$\mathrm{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi) = \theta^\chi_{L_{\chi,\infty}/k,S,T}(0)\Lambda_\chi,$$

where $\theta^\chi_{L_{\chi,\infty}/k,S,T}(0)$ is the χ-component of $\epsilon^{\emptyset}_{L_{\chi,\infty}/k,S,T}$, which is the Stickelberger element in this case. The above equality is nothing but the usual main conjecture proved by Wiles [36], so we have proved (i).

3.5.2. We now derive Theorem 3.21(ii) from the main result of Büyükboduk in [10]. To do this we assume condition $(**)$ and (without loss of generality) that K is totally real.

Set $r := [k : \mathbb{Q}] = \#S_\infty$. Since K is totally real, one has $V_\chi = S_\infty$ and $r_\chi = r$. By our assumptions (c) and (d), the χ-component of $\mathcal{X}_{L_{\chi,\infty},S\setminus S_\infty}$ vanishes. Therefore, the equality (13) becomes

$$\mathrm{char}_{\Lambda_\chi}((\bigcap^r U_{L_{\chi,\infty},S,T}/\langle\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}\rangle)^\chi) = \mathrm{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^\chi).$$

Since K is totally real and p is odd, we may assume that T is empty. Note that, since $L_{\chi,\infty}/L_\chi$ is the cyclotomic $\mathbb{Z}_p$-extension, the weak Leopoldt conjecture holds, and we have the canonical exact sequence

$$0 \to U_{L_{\chi,\infty}} \to U^{\mathrm{sl}}_{L_{\chi,\infty}} \to \mathrm{Gal}(M/L_{\chi,\infty}) \to A(L_{\chi,\infty}) \to 0, \tag{14}$$

where $U^{\mathrm{sl}}_{L_{\chi,\infty}}$ is the semi-local unit of $L_{\chi,\infty}$ at p, and M is the maximal abelian p-extension of $L_{\chi,\infty}$ unramified outside p. By our assumptions (c) and (d) again,

$U^{\chi}_{L_{\chi,\infty},S} = U^{\chi}_{L_{\chi,\infty}}$ and $A(L_{\chi,\infty})^{\chi} = A_S(L_{\chi,\infty})^{\chi}$. Therefore, what we have to prove is

$$\mathrm{char}_{\Lambda_\chi}((\bigwedge^r U^{\mathrm{sl}}_{L_{\chi,\infty}}/\langle \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S}\rangle)^{\chi}) = \mathrm{char}_{\Lambda_\chi}(\mathrm{Gal}(M/L_{\chi,\infty})^{\chi}).$$

This is nothing but [10, Theorem A]. Note that all of the hypotheses (a)-(f) occur as assumptions in the latter result. Indeed, (a) and (b) are (A1) and (A2) in [10] respectively, (c) is (A3) and the assumption on S in [10], and (d)-(f) are assumed in his main result. This completes the proof of Theorem 3.21(ii).

3.6. **Consequences for number fields of finite degree.** In this subsection we assume the condition $(**)$ stated at the beginning of §3.5 and also that K is a CM-field of finite degree over $\mathbb{Q}$. We shall describe unconditional results for K which follow the validity of Theorem 3.21(i).

To do this we set $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ and for any Λ-module M we denote by M^- the minus part consisting of elements on which the complex conjugation acts as -1 (namely, $M^- = e^-M$). We note, in particular, that $\theta_{K_\infty/k,S,T}(0)$ belongs to Λ^-.

We also write $x \mapsto x^{\#}$ for the $\mathbb{Z}_p$-linear involutions of both Λ and the group rings $\mathbb{Z}_p[G]$ for finite quotients G of $\mathrm{Gal}(K_\infty/k)$ which is induced by inverting elements of $\mathrm{Gal}(K_\infty/k)$.

Corollary 3.22. *If the p-adic μ-invariant of K_∞/K vanishes, then one has*

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)$$

and

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)^{\#}.$$

Proof. Since one has $r_\chi = 0$ for any odd character χ, the first displayed equality is equivalent to Conjecture 3.1 in this case and is therefore valid as a consequence of Theorem 3.21.

The second displayed equality is then obtained directly by applying the general result of [9, Lemma 2.8] to the first equality. □

Corollary 3.23. *Let L be an intermediate CM-field of K_∞/k which is finite over k, and set $G := \mathrm{Gal}(L/k)$. If the p-adic μ-invariant of K_∞/K vanishes, then there are equalities*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)$$

and

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)^{\#}.$$

Proof. This follows by combining Corollary 3.22 with the general result of Lemma 3.24 below and standard properties of Fitting ideals. □

Lemma 3.24. *Suppose that L/k is a Galois extension of finite number fields with Galois group G. Then there are natural isomorphisms*

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k})$$

and

$$\mathcal{S}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}_{S,T}(\mathbb{G}_{m/k}).$$

Proof. The 'Weil-étale cohomology complex' $R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ is perfect and so there exist projective $\mathbb{Z}[G]$-modules P_1 and P_2, and a homomorphism of G-modules $P_1 \to P_2$ whose cokernel identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})$ and is such that the cokernel of the induced map $P_1^G \to P_2^G$ identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k})$ (see [9, §5.4]).

The first isomorphism is then obtained by noting that the norm map induces an isomorphism of modules $(P_2)_G \xrightarrow{\sim} P_2^G$.

The second claimed isomorphism can also be obtained in a similar way, noting that $\mathcal{S}_{S,T}(\mathbb{G}_{m/L})$ is obtained as the cohomology in the highest (non-zero) degree of a perfect complex (see [9, Proposition 2.4]). □

We write $\mathcal{O}_L$ for the ring of integers of L and $\mathrm{Cl}^T(L)$ for the ray class group of $\mathcal{O}_L$ with modulus $\Pi_{w \in T_L} w$. We denote the Sylow p-subgroup of $\mathrm{Cl}^T(L)$ by $A^T(L)$ and write $(A^T(L)^-)^\vee$ for the Pontrjagin dual of the minus part of $A^T(L)$.

The next corollary of Theorem 3.21(i) that we record coincides with one of the main results of Greither and Popescu in [18].

Corollary 3.25. *Let L be an intermediate CM-field of K_∞/k which is finite over k, and set $G := \mathrm{Gal}(L/k)$. If the p-adic μ-invariant for K_∞/K vanishes, then one has*

$$\theta_{L/k,S,T}(0)^\# \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A^T(L)^-)^\vee).$$

Proof. The canonical exact sequence

$$0 \to \mathrm{Cl}^T(L)^\vee \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L}) \to \mathrm{Hom}(\mathcal{O}_L^\times, \mathbb{Z}) \to 0$$

from [9, Proposition 2.2] implies that the natural map

$$\mathcal{S}_{p,S_\infty(k),T}(\mathbb{G}_{m/L})^- \simeq (A^T(L)^-)^\vee$$

is bijective.

In addition, from [9, Proposition 2.4(ii)], we know that the canonical homomorphism

$$\mathcal{S}_{S,T}(\mathbb{G}_{m/L}) \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L})$$

is surjective.

The stated claim therefore follows directly from the second equality in Corollary 3.23. □

Remark 3.26.
(i) Our derivation of the equality in Corollary 3.25 differs from that given in [18] in that we avoid any use of the Galois modules related to 1-motives that are constructed in loc. cit.
(ii) The Brumer-Stark conjecture predicts $\theta_{L/k,S_{\mathrm{ram}}(L/k),T}(0)$ belongs to the annihilator $\mathrm{Ann}_{\mathbb{Z}_p[G]^-}(A^T(L))$ and if no p-adic place of L^+ splits in L, then Corollary 3.25 implies a stronger version of this conjecture.

We have assumed throughout §3 that the set S contains all p-adic places of k and so the Stickelberger element $\theta_{L/k,S,T}(0)$ in Corollary 3.25 can be imprimitive. In particular, if any p-adic prime of k splits completely in L, then $\theta_{L/k,S,T}(0)$ vanishes and the assertion of Corollary 3.25 is trivially valid.

However, by applying Corollary 1.2 in this context, we can now prove the following non-trivial result.

Corollary 3.27. *Let L be an intermediate CM-field of K_∞/k which is finite over k, and set $G := \mathrm{Gal}(L/k)$. If the p-adic μ-invariant for K_∞/K vanishes and at most one p-adic place of k splits in L/L^+, then one has*

$$\theta_{L/k,S_{\mathrm{ram}}(L/k),T}(0) \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A^T(L)^-)^\vee).$$

Proof. This follows immediately by combining [9, Corollary 1.14] with Corollary 1.2. □

4. Iwasawa-theoretic Rubin-Stark congruences

In this section, we formulate an Iwasawa-theoretic version of the conjecture proposed by Mazur and Rubin [24] and by the third author [28] (see also [9, Conjecture 5.4]). This conjecture is a natural generalization of the Gross-Stark conjecture [19], and plays a key role in the descent argument that we present in the next section.

We use the notation as in the previous section.

4.1. Statement of the congruences.

We first recall the formulation of the conjecture of Mazur and Rubin and of the third author.

Take a character $\chi \in \widehat{\mathcal{G}}$. Take a proper subset $V' \subset S$ so that all $v \in V'$ splits completely in L_χ (i.e. $\chi(G_v) = 1$) and that $V_\chi \subset V'$. Put $r' := \# V'$. We recall the formulation of the conjecture of Mazur and Rubin and of the third author for $(L_{\chi,n}/L_\chi/k, S, T, V_\chi, V')$. For simplicity, put

- $L_n := L_{\chi,n}$;
- $L := L_\chi$;
- $\mathcal{G}_n := \mathcal{G}_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/k)$;
- $G := G_\chi = \mathrm{Gal}(L_\chi/k)$;
- $\Gamma_n := \Gamma_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/L_\chi)$;
- $V := V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$;
- $r := r_\chi = \# V_\chi$.

Put $e := r' - r$. Let $I(\Gamma_n)$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_n]$. It is shown in [28, Lemma 2.11] that there exists a canonical injection

$$\bigcap^r U_{L,S,T} \hookrightarrow \bigcap^r U_{L_n,S,T}$$

which induces the injection

$$\nu_n : (\bigcap^r U_{L,S,T}) \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1} \hookrightarrow (\bigcap^r U_{L_n,S,T}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

Note that this injection does not coincides with the map induced by the inclusion $U_{L,S,T} \hookrightarrow U_{L_n,S,T}$, and we have

$$\nu_n(\mathrm{N}^r_{L_n/L}(a)) = \mathrm{N}_{L_n/L}\, a$$

for all $a \in \bigcap^r U_{L_n,S,T}$ (see [28, Remark 2.12]). Let I_n be the kernel of the natural map $\mathbb{Z}_p[\mathcal{G}_n] \to \mathbb{Z}_p[G]$. For $v \in V' \setminus V$, let $\mathrm{rec}_w : L^\times \to \Gamma_n$ denote the local reciprocity map at w (recall that w is the fixed place lying above v). Define

$$\mathrm{Rec}_w := \sum_{\sigma \in G} (\mathrm{rec}_w(\sigma(\cdot)) - 1)\sigma^{-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(L^\times, I_n/I_n^2).$$

It is shown in [28, Proposition 2.7] that $\bigwedge_{v \in V' \setminus V} \mathrm{Rec}_w$ induces a homomorphism

$$\mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1}.$$

Finally, define

$$\mathcal{N}_n : \bigcap^r U_{L_n,S,T} \to \bigcap^r U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}$$

by

$$\mathcal{N}_n(a) := \sum_{\sigma \in \Gamma_n} \sigma a \otimes \sigma^{-1}.$$

We now state the formulation of [28, Conjecture 3] (or [24, Conjecture 5.2]).

Conjecture 4.1 ($\mathrm{MRS}(L_n/L/k, S, T, V, V')_p$)**.** *Assume Conjectures* $\mathrm{RS}(L_n/k, S, T, V)_p$ *and* $\mathrm{RS}(L/k, S, T, V')_p$*. Then we have*

$$\mathcal{N}_n(\epsilon^V_{L_n/k,S,T}) = (-1)^{re}\nu_n(\mathrm{Rec}_n(\epsilon^{V'}_{L/k,S,T})) \ \textit{in} \ \bigcap^r U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

(Note that the sign in the right hand side depends on the labeling of S. We follow the convention in [9, §5.3]*.)*

Note that [9, Conjecture $\mathrm{MRS}(K/L/k, S, T, V, V')$] is slightly stronger than the above conjecture (see [9, Remark 5.7]).

We shall next give an Iwasawa theoretic version of the above conjecture. Note that, since the inverse limit $\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$ is isomorphic to $\mathbb{Z}_p$, the map

$$\varprojlim_n \mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

uniquely extends to give a $\mathbb{C}_p$-linear map

$$\mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1})$$

which we denote by Rec_∞.

Conjecture 4.2 ($\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$)**.** *Assume that Conjecture* $\mathrm{RS}(L_n/k, S, T, V)_p$ *is valid for all n. Then, there exists a (unique)*

$$\kappa = (\kappa_n)_n \in \bigcap^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

such that

$$\nu_n(\kappa_n) = \mathcal{N}_n(\epsilon^V_{L_n/k,S,T})$$

for all n and that

$$e_\chi \kappa = (-1)^{re} e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}) \ \textit{in}\ \mathbb{C}_p(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}).$$

Remark 4.3. Clearly the validity of Conjecture $\mathrm{MRS}(L_n/L/k, S, T, V, V')_p$ for all n implies the validity of $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$. A significant advantage of the above formulation of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is that we do not need to assume that Conjecture $\mathrm{RS}(L/k, S, T, V')_p$ is valid.

Proposition 4.4.

(i) *If $V = V'$, then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*
(ii) *If $V \subset V'' \subset V'$, then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'')$.
(iii) *Suppose that $\chi(G_v) = 1$ for all $v \in S$ and $\#V' = \#S - 1$. Then, for any $V'' \subset S$ with $V \subset V''$ and $\#V'' = \#S - 1$,* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *and* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'')$ *are equivalent.*
(iv) *If $v \in V' \setminus V$ is a finite place which is unramified in L_∞, then* $\mathrm{MRS}(K_\infty/k, S \setminus \{v\}, T, \chi, V' \setminus \{v\})$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.
(v) *If $\#V' \neq \#S - 1$ and $v \in S \setminus V'$ is a finite place which is unramified in L_∞, then* $\mathrm{MRS}(K_\infty/k, S \setminus \{v\}, T, \chi, V')$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.

Proof. Claim (i) follows from the 'norm relation' of Rubin-Stark elements, see [28, Remark 3.9] or [24, Proposition 5.7]. Claim (ii) follows from [28, Proposition 3.12]. Claim (iii) follows from [29, Lemma 5.1]. Claim (iv) follows from the proof of [28, Proposition 3.13]. Claim (v) follows by noting that

$$\epsilon^V_{L_n/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^V_{L_n/k,S\setminus\{v\},T}$$

and

$$\epsilon^{V'}_{L/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^{V'}_{L/k,S\setminus\{v\},T}.$$

□

Corollary 4.5. *If every place v in $V' \setminus V$ is both non-archimedean and unramified in L_∞, then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

Proof. By Proposition 4.4(iv), we may assume $V = V'$. By Proposition 4.4(i), $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is valid in this case. □

Consider the following condition:

$$\mathrm{NTZ}(K_\infty/k, \chi) \quad \chi(G_\mathfrak{p}) \neq 1 \text{ for all } \mathfrak{p} \in S_p(k) \text{ which ramify in } L_{\chi,\infty}.$$

This condition is usually called 'no trivial zeros'.

Corollary 4.6. *Assume that χ satisfies* $\mathrm{NTZ}(K_\infty/k, \chi)$*. Then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

Proof. In this case we see that every $v \in V' \setminus V$ is finite and unramified in L_∞. □

4.2. **Connection to the Gross-Stark conjecture.** In this subsection we help set the context for Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ by showing that it specializes to recover the Gross-Stark Conjecture (as stated in Conjecture 4.7 below).

To do this we assume throughout that k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension and χ is totally odd. We also set $V' := \{v \in S \mid \chi(G_v) = 1\}$ (and note that this is a proper subset of S since χ is totally odd) and we assume that every $v \in V'$ lies above p (noting that this assumption is not restrictive as a consequence of Proposition 4.4(iv)).

We shall now show that this case of $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is equivalent to the Gross-Stark conjecture.

As a first step, we note that in this case V is empty (that is, $r = 0$) and so one knows that Conjecture $\mathrm{RS}(L_n/k, S, T, V)_p$ is valid for all n (by [27, Theorem 3.3]). In fact, one has $\epsilon^V_{L_n/k,S,T} = \theta_{L_n/k,S,T}(0) \in \mathbb{Z}_p[\mathcal{G}_n]$ and, by [24, Proposition 5.4], the assertion of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is equivalent to the following claims: one has

$$\theta_{L_n/k,S,T}(0) \in I_n^{r'} \tag{15}$$

for all n and

$$e_\chi \theta_{L_\infty/k,S,T}(0) = e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}) \text{ in } \mathbb{C}_p[G] \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}, \tag{16}$$

where we set

$$\theta_{L_\infty/k,S,T}(0) := \varprojlim_n \theta_{L_n/k,S,T}(0) \in \varprojlim_n I_n^{r'}/I_n^{r'+1} \simeq \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}.$$

We also note that the validity of (15) follows as a consequence of our Iwasawa main conjecture (Conjecture 3.1) by using Proposition 2.6(iii) and the result of [9, Lemma 5.19] (see the argument in §5.3).

To study (16) we set $\chi_1 := \chi|_\Delta \in \widehat{\Delta}$ and regard (as we may) the product $\chi_2 := \chi\chi_1^{-1}$ as a character of $\Gamma = \mathrm{Gal}(k_\infty/k)$.

Note that $\mathrm{Gal}(L_\infty/k) = G_{\chi_1} \times \Gamma_{\chi_1}$. Fix a topological generator $\gamma \in \Gamma_{\chi_1}$, and identify $\mathbb{Z}_p[\mathrm{im}(\chi_1)][[\Gamma_{\chi_1}]]$ with the ring of power series $\mathbb{Z}_p[\mathrm{im}(\chi_1)][[t]]$ via the correspondence $\gamma = 1 + t$.

We then define $g^{\chi_1}_{L_\infty/k,S,T}(t)$ to be the image of $\theta_{L_\infty/k,S,T}(0)$ under the map

$$\mathbb{Z}_p[[\mathrm{Gal}(L_\infty/k)]] = \mathbb{Z}_p[G_{\chi_1}][[\Gamma_{\chi_1}]] \to \mathbb{Z}_p[\mathrm{im}(\chi_1)][[\Gamma_{\chi_1}]] = \mathbb{Z}_p[\mathrm{im}(\chi_1)][[t]]$$

induced by χ_1. We recall that the p-adic L-function of Deligne-Ribet is defined by

$$L_{k,S,T,p}(\chi^{-1}\omega, s) := g^{\chi_1}_{L_\infty/k,S,T}(\chi_2(\gamma)\chi_{\mathrm{cyc}}(\gamma)^s - 1),$$

where χ_{cyc} is the cyclotomic character, and we note that one can show $L_{k,S,T,p}(\chi^{-1}\omega, s)$ to be independent of the choice of γ.

The validity of (15) implies an inequality

$$\mathrm{ord}_{s=0} L_{k,S,T,p}(\chi^{-1}\omega, s) \geq r'. \tag{17}$$

It is known that (17) is a consequence of the Iwasawa main conjecture (in the sense of Wiles [36]), which is itself known to be valid when p is odd. In addition, Spiess has recently proved that (17) is valid, including the case $p = 2$, by using Shintani cocycles [32]. In all cases, therefore, we can define

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) := \lim_{s\to 0} s^{-r'} L_{k,S,T,p}(\chi^{-1}\omega, s) \in \mathbb{C}_p.$$

For $v \in V'$, define

$$\mathrm{Log}_w : L^\times \to \mathbb{Z}_p[G]$$

by

$$\mathrm{Log}_w(a) := -\sum_{\sigma\in G} \log_p(\mathrm{N}_{L_w/\mathbb{Q}_p}(\sigma a))\sigma^{-1},$$

where $\log_p : \mathbb{Q}_p^\times \to \mathbb{Z}_p$ is Iwasawa's logarithm (in the sense that $\log_p(p) = 0$). We set

$$\mathrm{Log}_{V'} := \bigwedge_{v\in V'} \mathrm{Log}_w : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

We shall denote the map $\mathbb{C}_p[G] \to \mathbb{C}_p$ induced by χ also by χ.

For $v \in V'$, we define

$$\mathrm{Ord}_w : L^\times \to \mathbb{Z}[G]$$

by

$$\mathrm{Ord}_w(a) := \sum_{\sigma\in G} \mathrm{ord}_w(\sigma a)\sigma^{-1},$$

and set

$$\mathrm{Ord}_{V'} := \bigwedge_{v\in V'} \mathrm{Ord}_w : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

On the χ-component, $\mathrm{Ord}_{V'}$ induces an isomorphism

$$\chi \circ \mathrm{Ord}_{V'} : e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \xrightarrow{\sim} \mathbb{C}_p.$$

Taking a non-zero element $x \in e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$, we define the $\mathcal{L}$-invariant by

$$\mathcal{L}(\chi) := \frac{\chi(\mathrm{Log}_{V'}(x))}{\chi(\mathrm{Ord}_{V'}(x))} \in \mathbb{C}_p.$$

Since $e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$ is a one dimensional $\mathbb{C}_p$-vector space, we see that $\mathcal{L}(\chi)$ does not depend on the choice of x.

Then the Gross-Stark conjecture is stated as follows.

Conjecture 4.7 ($\mathrm{GS}(L/k, S, T, \chi)$)**.**

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

Remark 4.8. This formulation constitutes a natural higher rank generalization of the form of the Gross-Stark conjecture that is considered by Darmon, Dasgupta and Pollack (see [12, Conjecture 1]).

Letting $x = e_\chi \epsilon^{V'}_{L/k,S,T}$, we obtain

$$\chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

Thus we see that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to the equality

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})).$$

Concerning the relation between Rec_∞ and $\mathrm{Log}_{V'}$, we note the fact

$$\chi_{\mathrm{cyc}}(\mathrm{rec}_w(a)) = \mathrm{N}_{L_w/\mathbb{Q}_p}(a)^{-1},$$

where $v \in V'$ and $a \in L^\times$.

Given this fact, it is straightforward to check (under the validity of (15)) that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to (16).

At this stage we have therefore proved the following result.

Theorem 4.9. *Suppose that k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and χ is totally odd. Set $V' := \{v \in S \mid \chi(G_v) = 1\}$ and assume that every $v \in V'$ lies above p. Assume also that (15) is valid. Then Conjecture* $\mathrm{GS}(L/k, S, T, \chi)$ *is equivalent to Conjecture* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.

4.3. **A proof in the case $k = \mathbb{Q}$.** In [9, Corollary 1.2] the known validity of the eTNC for Tate motives over abelian fields is used to prove that Conjecture $\mathrm{MRS}(K/L/k, S, T, V, V')$ is valid in the case $k = \mathbb{Q}$.

In this subsection, we shall give a much simpler proof of the latter result which uses only Theorem 4.9, the known validity of the Gross-Stark conjecture over abelian fields and a classical result of Solomon [30].

We note that for any χ and n the Rubin-Stark conjecture is known to be true for $(L_{\chi,n}/\mathbb{Q}, S, T, V_\chi)$. (In this setting the Rubin-Stark element is given by a cyclotomic unit (resp. the Stickelberger element) when $r_\chi = 1$ (resp. $r_\chi = 0$).)

Theorem 4.10. *Suppose that $k = \mathbb{Q}$. Then,* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

Proof. By Proposition 4.4(ii), we may assume that V' is maximal, namely,

$$r' = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}.$$

By Corollary 4.6, we may assume that $\chi(p) = 1$.

Suppose first that χ is odd. Since Conjecture $\mathrm{GS}(L/\mathbb{Q}, S, T, \chi)$ is valid (see [19, §4]), Conjecture $\mathrm{MRS}(K_\infty/\mathbb{Q}, S, T, \chi, V')$ follows from Theorem 4.9.

Suppose next that $\chi = 1$. In this case we have $r' = \#S - 1$. We may assume $p \notin V'$ by Proposition 4.4(iii). In this case every $v \in V' \setminus V$ is unramified in L_∞. Hence, the theorem follows from Corollary 4.5.

Finally, suppose that $\chi \neq 1$ is even. By Proposition 4.4(iv) and (v), we may assume

$$S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q}) \text{ and } V' = \{\infty, p\}.$$

We label $S = \{v_0, v_1, \ldots\}$ so that $v_1 = \infty$ and $v_2 = p$.

Fix a topological generator γ of $\Gamma = \mathrm{Gal}(L_\infty/L)$. Then we construct an element $\kappa(L, \gamma) \in \varprojlim_n L^\times/(L^\times)^{p^n}$ as follows. Note that $\mathrm{N}_{L_n/L}(\epsilon^V_{L_n/\mathbb{Q},S,T})$ vanishes since $\chi(p) = 1$. So we can take $\beta_n \in L_n^\times$ such that $\beta_n^{\gamma-1} = \epsilon^V_{L_n/\mathbb{Q},S,T}$ (Hilbert's theorem 90). Define

$$\kappa_n := \mathrm{N}_{L_n/L}(\beta_n) \in L^\times/(L^\times)^{p^n}.$$

This element is independent of the choice of β_n, and for any $m > n$ the natural map

$$L^\times/(L^\times)^{p^m} \to L^\times/(L^\times)^{p^n}$$

sends κ_m to κ_n. We define

$$\kappa(L, \gamma) := (\kappa_n)_n \in \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Then, by Solomon [30, Proposition 2.3(i)], we know that

$$\kappa(L, \gamma) \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L\left[\frac{1}{p}\right]^\times \hookrightarrow \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Fix a prime $\mathfrak{p}$ of L lying above p. Define

$$\mathrm{Ord}_{\mathfrak{p}} : L^\times \to \mathbb{Z}_p[G]$$

by $\mathrm{Ord}_{\mathfrak{p}}(a) := \sum_{\sigma\in G}\mathrm{ord}_{\mathfrak{p}}(\sigma a)\sigma^{-1}$. Similarly, define

$$\mathrm{Log}_{\mathfrak{p}} : L^{\times} \to \mathbb{Z}_p[G]$$

by $\mathrm{Log}_{\mathfrak{p}}(a) := -\sum_{\sigma\in G}\log_p(\iota_{\mathfrak{p}}(\sigma a))\sigma^{-1}$, where $\iota_{\mathfrak{p}} : L \hookrightarrow L_{\mathfrak{p}} = \mathbb{Q}_p$ is the natural embedding.

Then by the result of Solomon [30, Theorem 2.1 and Remark 2.4], one deduces

$$\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)) = -\frac{1}{\log_p(\chi_{\mathrm{cyc}}(\gamma))}\mathrm{Log}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}).$$

From this, we have

(18) $\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)) \otimes (\gamma-1) = -\mathrm{Rec}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T})$ in $\mathbb{Z}_p[G]\otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2$,

where $I(\Gamma)$ is the augmentation ideal of $\mathbb{Z}_p[[\Gamma]]$.

We know that $e_\chi\mathbb{C}_pU_{L,S}$ is a two-dimensional $\mathbb{C}_p$-vector space. Lemma 4.11 below shows that $\{e_\chi\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}, e_\chi\kappa(L,\gamma)\}$ is a $\mathbb{C}_p$-basis of this space. For simplicity, set $\epsilon^V_L := \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}$. Note that the isomorphism

$$\mathrm{Ord}_{\mathfrak{p}} : e_\chi\mathbb{C}_p\bigwedge^2 U_{L,S} \xrightarrow{\sim} e_\chi\mathbb{C}_pU_L$$

sends $e_\chi\epsilon^V_L \wedge \kappa(L,\gamma)$ to $-\chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))e_\chi\epsilon^V_L$. Since we have

$$\mathrm{Ord}_{\mathfrak{p}}(e_\chi\epsilon^{V'}_{L/\mathbb{Q},S,T}) = -e_\chi\epsilon^V_L$$

(see [27, Proposition 5.2] or [28, Proposition 3.6]), we have

$$e_\chi\epsilon^{V'}_{L/\mathbb{Q},S,T} = -\chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))^{-1}e_\chi\epsilon^V_L \wedge \kappa(L,\gamma).$$

Hence we have

$$\begin{aligned}\mathrm{Rec}_{\mathfrak{p}}(e_\chi\epsilon^{V'}_{L/\mathbb{Q},S,T}) &= \chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))^{-1}e_\chi\kappa(L,\gamma)\cdot\mathrm{Rec}_{\mathfrak{p}}(\epsilon^V_L)\\ &= -e_\chi\kappa(L,\gamma)\otimes(\gamma-1),\end{aligned}$$

where the first equality follows by noting that $\mathrm{Rec}_{\mathfrak{p}}(\kappa(L,\gamma)) = 0$ (since $\kappa(L,\gamma)$ lies in the universal norm by definition), and the second by (18).

Now, noting that

$$\nu_n : U_{L,S,T}\otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 \hookrightarrow U_{L_n,S,T}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^2$$

is induced by the inclusion map $L \hookrightarrow L_n$, and that

$$\mathcal{N}_n(\epsilon^V_{L_n/\mathbb{Q},S,T}) = \kappa_n \otimes (\gamma-1),$$

it is easy to see that the element $\kappa := \kappa(L,\gamma)\otimes(\gamma-1)$ has the properties in the statement of Conjecture $\mathrm{MRS}(K_\infty/\mathbb{Q},S,T,\chi,V')$.

This completes the proof the claimed result. $\square$

Lemma 4.11. *Assume that $k = \mathbb{Q}$ and $\chi \neq 1$ is even such that $\chi(p) = 1$. Assume also that $S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q})$. Then, $\{e_\chi \epsilon^V_{L/\mathbb{Q}, S\setminus\{p\}, T}, e_\chi \kappa(L, \gamma)\}$ is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L,S}$.*

Proof. This result follows from [31, Remark 4.4]. But we give a sketch of another proof, which is essentially given by Flach in [14].

In the next section, we define the 'Bockstein map'

$$\beta : e_\chi \mathbb{C}_p U_{L,S} \to e_\chi \mathbb{C}_p(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2).$$

We see that β is injective on $e_\chi \mathbb{C}_p U_L$, and that

$$\ker \beta \simeq U_{L_\infty, S} \otimes_\Lambda \mathbb{C}_p,$$

where we put $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$ and $\mathbb{C}_p$ is regarded as a Λ-algebra via χ. Hence we have

$$e_\chi \mathbb{C}_p U_{L,S} = e_\chi \mathbb{C}_p U_L \oplus (U_{L_\infty, S} \otimes_\Lambda \mathbb{C}_p).$$

Since $e_\chi \epsilon^V_{L/\mathbb{Q}, S\setminus\{p\}, T}$ is non-zero, this is a basis of $e_\chi \mathbb{C}_p U_{L, S\setminus\{p\}} = e_\chi \mathbb{C}_p U_L$. We prove that $e_\chi \kappa(L, \gamma)$ is a basis of $U_{L_\infty, S} \otimes_\Lambda \mathbb{C}_p$.

By using the exact sequence

$$0 \to U_{L_\infty, S} \xrightarrow{\gamma - 1} U_{L_\infty, S} \to U_{L,S},$$

we see that there exists a unique element $\alpha \in U_{L_\infty, S}$ such that $(\gamma - 1)\alpha = \epsilon^V_{L_\infty/\mathbb{Q}, S, T}$. By the cyclotomic Iwasawa main conjecture over $\mathbb{Q}$, we see that α is a basis of $U_{L_\infty, S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi}$, where $\mathfrak{p}_\chi := \ker(\chi : \Lambda \to \mathbb{C}_p)$. The image of α under the map

$$U_{L_\infty, S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi} \xrightarrow{\chi} U_{L_\infty, S} \otimes_\Lambda \mathbb{C}_p \hookrightarrow e_\chi \mathbb{C}_p U_{L,S}$$

is equal to $e_\chi \kappa(L, \gamma)$. □

5. A strategy for proving the eTNC

5.1. **Statement of the main result and applications.** In the sequel we fix an intermediate field L of K_∞/k which is finite over k and set $G := \mathrm{Gal}(L/k)$. In this section we always assume the following conditions to be satisfied:

(R) for every $\chi \in \widehat{G}$, one has $r_{\chi,S} < \#S$;
(S) no finite place of k splits completely in k_∞.

Remark 5.1. Before proceeding we note that condition (R) is very mild since it is automatically satisfied when the class number of k is equal to one and, for any k, is satisfied when S is large enough. We also note that condition (S) is satisfied when, for example, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension.

The following result is one of the main results of this article and, as we will see, it provides an effective strategy for proving the special case of the eTNC that we are considering here.

Theorem 5.2. *Assume the following conditions:*

(hIMC) *The main conjecture* $\mathrm{IMC}(K_\infty/k, S, T)$ *is valid;*

(F) *for every* χ *in* $\widehat{G}$*, the module of* Γ_χ*-coinvariants of* $A_S^T(L_{\chi,\infty})$ *is finite;*

(MRS) *for every* χ *in* $\widehat{G}$*, Conjecture* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ *is valid for a maximal set* V'_χ *(so that* $\#V'_\chi = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}$*).*

Then, the conjecture $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G])$ *is valid.*

Remark 5.3. We note that the set V'_χ in condition (MRS) is not uniquely determined when every place v in S satisfies $\chi(G_v) = 1$, but that the validity of the conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ is independent of the choice of V'_χ (by Proposition 4.4(iii)).

Remark 5.4. One checks easily that the condition (F) is equivalent to the finiteness of the the module of Γ_χ-coinvariants of $A_S(L_{\chi,\infty})$. Hence, taking account of an observation of Kolster in [23, Theorem 1.14], condition (F) can be regarded as a natural generalization of the Gross conjecture [19, Conjecture 1.15]. In particular, we recall that condition (F) is satisfied in each of the following cases:

- L is abelian over $\mathbb{Q}$ (due to Greenberg, see [17]),
- k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension and L has unique p-adic place (in this case '$\delta_L = 0$' holds obviously, see [23]),
- L is totally real and the Leopoldt conjecture is valid for L at p (see [23, Corollary 1.3]).

Remark 5.5. The condition (MRS) is satisfied for χ in $\widehat{G}$ when the condition $\mathrm{NTZ}(K_\infty/k, \chi)$ is satisfied (see Corollary 4.6).

As an immediate corollary of Theorem 5.2, we obtain a new proof of a theorem that was first proved by Greither and the first author [8] for p odd, and by Flach [15] for $p = 2$.

Corollary 5.6. *If* $k = \mathbb{Q}$*, then the conjecture* $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G])$ *is valid.*

Proof. As we mentioned above, the conditions (R), (S) and (F) are all satisfied in this case. In addition, the condition (hIMC) is a direct consequence of the classical Iwasawa main conjecture solved by Mazur and Wiles (see [8] and [15]) and the condition (MRS) is satisfied by Theorem 4.10. □

We also obtain a result over totally real fields.

Corollary 5.7. *Suppose that* p *is odd,* k *is totally real,* k_∞/k *is the cyclotomic* $\mathbb{Z}_p$*-extension, and* K *is CM. Assume that (F) is satisfied, that the* μ*-invariant of* K_∞/K *vanishes, and that for every odd character* $\chi \in \widehat{G}$ *Conjecture* $\mathrm{GS}(L_\chi/k, S, T, \chi)$ *is valid. Then, Conjecture* $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]^-)$ *is valid.*

Proof. We take S so that condition (R) is satisfied. Then the minus-part of condition (hIMC) is satisfied by Theorem 3.21(i) and the minus part of condition (MRS) by Theorem 4.2. □

When at most one p-adic place $\mathfrak{p}$ of k satisfies $\chi(G_{\mathfrak{p}}) = 1$, Dasgupta, Darmon and Pollack proved the validity of Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ under some assumptions including Leopoldt's conjecture (see [12]). Recently in the same case Ventullo asserts in [35] that Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ is unconditionally valid. In this case condition (F) is also valid by the argument of Gross in [19, Proposition 2.13]. Hence we get the following

Corollary 5.8. *Suppose that p is odd, k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and K is CM. Assume that the μ-invariant of K_∞/K vanishes, and that for each odd character $\chi \in \widehat{G}$ there is at most one p-adic place $\mathfrak{p}$ of k which satisfies $\chi(G_{\mathfrak{p}}) = 1$. Then, Conjecture* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-)$ *is valid.*

Examples 5.9. It is not difficult to find many concrete families of examples which satisfy all of the hypotheses of Corollary 5.8 and hence to deduce the validity of $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-)$ in some new and interesting cases. In particular, we shall now describe several families of examples in which the extension $k/\mathbb{Q}$ is not abelian (noting that if $L/\mathbb{Q}$ is abelian and $k \subset L$, then $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is already known to be valid).

(i) The case $p = 3$. As a simple example, we consider the case that $k/\mathbb{Q}$ is a S_3-extension. To do this we fix an irreducible cubic polynomial $f(x)$ in $\mathbb{Z}[x]$ with discriminant $27d$ where d is strictly positive and congruent to 2 modulo 3. (For example, one can take $f(x)$ to be $x^3 - 6x - 3$, $x^3 - 15x - 3$, etc.) The minimal splitting field k of $f(x)$ over $\mathbb{Q}$ is then totally real (since $27d > 0$) and an S_3-extension of $\mathbb{Q}$ (since $27d$ is not a square). Also, since the discriminant of $f(x)$ is divisible by 27 but not 81, the prime 3 is totally ramified in k. Now set $p := 3$ and $K := k(\mu_p) = k(\sqrt{-p}) = k(\sqrt{-d})$. Then the prime above p splits in K/k because $-d \equiv 1 \pmod 3$. In addition, as $K/\mathbb{Q}(\sqrt{d}, \sqrt{-p})$ is a cyclic cubic extension, the μ-invariant of K_∞/K vanishes and so the extension K/k satisfies all the conditions of Corollary 5.8 (with $p = 3$).

(ii) The case $p > 3$. In this case one can construct a suitable field K in the following way. Fix a primitive p-th root of unity ζ, an integer i such that $1 \leq i \leq (p-3)/2$ and an integer b which is prime to p, and then set

$$a := (1 + b(\zeta - 1))^{2i+1}/(1 + b(\zeta^{-1} - 1)^{2i+1}).$$

Write ord_π for the normalized additive valuation of $\mathbb{Q}(\mu_p)$ associated to the prime element $\pi = \zeta - 1$. Then, since $\mathrm{ord}_\pi(a - 1) = 2i + 1 < p$, (π) is totally ramified in $\mathbb{Q}(\mu_p, \sqrt[p]{a})/\mathbb{Q}(\mu_p)$. Also, since $\rho(a) = a^{-1}$ where ρ is the complex conjugation, $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is the composite of a cyclic extension of $\mathbb{Q}(\mu_p)^+$ of degree p and $\mathbb{Q}(\mu_p)$. This shows that $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is a CM-field and, since $1 < 2i + 1 < p$, the extension $\mathbb{Q}(\mu_p, \sqrt[p]{a})^+/\mathbb{Q}$ is non-abelian. We now take a negative integer $-d$ which is a quadratic residue modulo p, let K denote the CM-field $\mathbb{Q}(\mu_p, \sqrt[p]{a}, \sqrt{-d})$ and set $k := K^+$. Then p is totally ramified in $k/\mathbb{Q}$ and the p-adic prime of k splits in K. In addition, $k/\mathbb{Q}$ is not abelian and the μ-invariant of K_∞/K vanishes since $K/\mathbb{Q}(\mu_p, \sqrt{-d})$ is cyclic

of degree p. This shows that the extension K/k satisfies all of the hypotheses of Corollary 5.8.
(iii) In both of the cases (i) and (ii) described above, p is totally ramified in the extension $k_\infty/\mathbb{Q}$ and so Corollary 5.8 implies that $\mathrm{eTNC}(h^0(\operatorname{Spec} K_n), \mathbb{Z}_p[G]^-)$ is valid for any non-negative integer n. In addition, if F is any real abelian field of degree prime to $[k:\mathbb{Q}]$ in which p is totally ramified, the minus component of the p-part of eTNC for FK_n/k holds for any non-negative integer n.

Remark 5.10. Finally we note that, by using similar methods to the proofs of the above corollaries it is also possible to deduce the main result of Bley [2] as a consequence of Theorem 5.2. In this case k is imaginary quadratic, the validity of (hIMC) can be derived from Rubin's result in [26] (as explained in [2]), and the conjecture (MRS) from Bley's result [1], which is itself an analogue of Solomon's theorem [30] for elliptic units, by using the same argument as Theorem 4.10.

5.2. **A computation of Bockstein maps.** Fix a character $\chi \in \widehat{G}$. For simplicity, we set

- $L_n := L_{\chi,n}$;
- $L := L_\chi$;
- $V := V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$;
- $r := r_\chi = \# V_\chi$;
- $V' := V'_\chi$ (as in (MRS) in Theorem 5.2);
- $r' := r_{\chi,S} = \# V'$;
- $e := r' - r$.

As in §4.1, we label $S = \{v_0, v_1, \ldots\}$ so that $V = \{v_1, \ldots, v_r\}$ and $V' = \{v_1, \ldots, v_{r'}\}$, and fix a place w lying above each $v \in S$. Also, as in §2.4, it will be useful to fix a representative of $C_{K_\infty,S,T}$:

$$\Pi_{K_\infty} \to \Pi_{K_\infty},$$

where the first term is placed in degree zero, and Π_{K_∞} is a free Λ-module with basis $\{b_1, \ldots, b_d\}$. This representative is chosen so that the natural surjection

$$\Pi_{K_\infty} \to H^1(C_{K_\infty,S,T}) \to \mathcal{X}_{K_\infty,S}$$

sends b_i to $w_i - w_0$ for every i with $1 \le i \le r'$.

We define a height one regular prime ideal of Λ by setting

$$\mathfrak{p} := \ker(\Lambda \xrightarrow{\chi} \mathbb{Q}_p(\chi) := \mathbb{Q}_p(\operatorname{im}\chi)).$$

Then the localization $R := \Lambda_\mathfrak{p}$ is a discrete valuation ring and we write P for its maximal ideal. We see that χ induces an isomorphism

$$E := R/P \xrightarrow{\sim} \mathbb{Q}_p(\chi).$$

We set $C := C_{K_\infty,S,T} \otimes_\Lambda R$ and $\Pi := \Pi_{K_\infty} \otimes_\Lambda R$.

Lemma 5.11. *Let γ be a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Let n be an integer which satisfies $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$. Then $\gamma^{p^n} - 1$ is a uniformizer of R.*

Proof. Regard $\chi \in \widehat{\mathcal{G}}$, and put $\chi_1 := \chi|_\Delta \in \widehat{\Delta}$. We identify R with the localization of $\Lambda_{\chi_1}[1/p] = \mathbb{Z}_p[\mathrm{im}\,\chi_1][[\Gamma]][1/p]$ at $\mathfrak{q} := \ker(\Lambda_{\chi_1}[1/p] \stackrel{\chi|_\Gamma}{\to} \mathbb{Q}_p(\chi))$.

Then the lemma follows by noting that the localization of $\Lambda_{\chi_1}[1/p]/(\gamma^{p^n} - 1) = \mathbb{Z}_p[\mathrm{im}\,\chi_1][\Gamma_n][1/p]$ at $\mathfrak{q}$ is identified with $\mathbb{Q}_p(\chi)$. □

Lemma 5.12. *Assume that the condition (F) is satisfied.*

(i) *$H^0(C)$ is isomorphic to $U_{K_\infty,S,T} \otimes_\Lambda R$, and R-free of rank r.*
(ii) *$H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S} \otimes_\Lambda R$.*
(iii) *The maximal R-torsion submodule $H^1(C)_{\mathrm{tors}}$ of $H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, and annihilated by P. (So $H^1(C)_{\mathrm{tors}}$ is an E-vector space.)*
(iv) *$H^1(C)_{\mathrm{tf}} := H^1(C)/H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{Y}_{K_\infty,V} \otimes_\Lambda R$ and is therefore R-free of rank r.*
(v) $\dim_E(H^1(C)_{\mathrm{tors}}) = e$.

Proof. Since $U_{K_\infty,S,T} \otimes_\Lambda R = H^0(C)$ is regarded as a submodule of Π, we see that $U_{K_\infty,S,T} \otimes_\Lambda R$ is R-free. Put $\chi_1 := \chi|_\Delta \in \widehat{\Delta}$. Note that $L_\infty := L_{\chi,\infty} = L_{\chi_1,\infty}$, and that the quotient field of R is $Q(\Lambda_{\chi_1})$. As in the proof of Theorem 3.5, we have

$$U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{L_\infty,V} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_{\chi_1}).$$

These are r-dimensional $Q(\Lambda_{\chi_1})$-vector spaces. This proves (i).

To prove (ii), it is sufficient to show that $A_S^T(K_\infty) \otimes_\Lambda R = 0$. Fix a topological generator γ of Γ, and regard $\mathbb{Z}_p[[\Gamma]]$ as the ring of power series $\mathbb{Z}_p[[T]]$ via the identification $\gamma = 1 + T$. Let f be the characteristic polynomial of the $\mathbb{Z}_p[[T]]$-module $A_S^T(L_\infty)$. By Lemma 5.11, for sufficiently large n, $\gamma^{p^n} - 1$ is a uniformizer of R. On the other hand, by the assumption (F), we see that f is prime to $\gamma^{p^n} - 1$. This implies (ii).

We prove (iii). Proving that $H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, it is sufficient to show that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{K_\infty,V} \otimes_\Lambda Q(\Lambda_{\chi_1}),$$

by (ii). This has been shown in the proof of Theorem 3.5. We prove that $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$ is annihilated by P. Note that

$$\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R = \mathcal{X}_{K_\infty,S\setminus(V\cup S_\infty)} \otimes_\Lambda R,$$

since the complex conjugation c at $v \in S_\infty \setminus (V \cap S_\infty)$ is non-trivial in G_{χ_1}, and hence $c - 1 \in R^\times$. Hence, it is sufficient to show that, for every $v \in S \setminus (V \cup S_\infty)$, there exists $\sigma \in G_v \cap \Gamma$ such that $\sigma - 1$ is a uniformizer of R, where $G_v \subset \mathcal{G}$ is the decomposition group at a place of K_∞ lying above v. Thanks to the assumption (S), we find such σ by Lemma 5.11.

The assertion (iv) is immediate from the above argument.

The assertion (v) follows from (iii), (iv), and that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda E \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{Y}_{L,V'}$$

is an r'-dimensional E-vector space. □

In the following for any R-module M we often denote $M \otimes_R E$ by M_E. Also, we assume that (F) is satisfied.

Definition 5.13. The 'Bockstein map' is the map

$$\begin{aligned} \beta : H^0(C_E) &\to H^1(C \otimes_R P) \\ &= H^1(C) \otimes_R P \\ &\to H^1(C_E) \otimes_E P/P^2 \end{aligned}$$

induced by the exact triangle

$$C \otimes_R P \to C \to C_E.$$

Note that there are canonical isomorphisms

$$H^0(C_E) \simeq U_{L,S,T} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi) U_{L,S,T},$$
$$H^1(C_E) \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{Y}_{L,V'},$$

where $\mathbb{Q}_p(\chi)$ is regarded as a $\mathbb{Z}_p[G_\chi]$-algebra via χ. Note also that P is generated by $\gamma^{p^n} - 1$ with sufficiently large n, where γ is a fixed topological generator of Γ (see Lemma 5.11). There is a canonical isomorphism

$$I(\Gamma_\chi)/I(\Gamma_\chi)^2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\chi) \simeq P/P^2,$$

where $I(\Gamma_\chi)$ denotes the augmentation ideal of $\mathbb{Z}_p[[\Gamma_\chi]]$. (Note that $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Gamma_\chi = \mathrm{Gal}(L_\infty/L)$.) Thus, the Bockstein map is regarded as the map

$$\beta : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \simeq e_\chi \mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2).$$

Proposition 5.14. *The Bockstein map β is induced by the map*

$$U_{L,S,T} \to \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2$$

given by $a \mapsto \sum_{w \in S_L} w \otimes (\mathrm{rec}_w(a) - 1)$.

Proof. The proof is the same as that of [14, Lemma 5.8]. We sketch the proof given in loc. cit.

Take n so that the image of $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$ in $\mathrm{Gal}(L_\infty/L) = \Gamma_\chi$ is a generator. We regard $\gamma^{p^n} \in \Gamma_\chi$. Define $\theta \in H^1(L, \mathbb{Z}_p) = \mathrm{Hom}(G_L, \mathbb{Z}_p)$ by $\gamma^{p^n} \mapsto 1$. Define

$$\beta' : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \xrightarrow{\sim} e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S}$$

by $\beta(a) = \beta'(a) \otimes (\gamma^{p^n} - 1)$. Then, β' is induced by the cup product

$$\cdot \cup \theta : \mathbb{Q}_p U_{L,S} \simeq H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \to H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p \mathcal{X}_{L,S \setminus S_\infty}.$$

By class field theory we see that β is induced by the map $a \mapsto \sum_{w \in S_L \setminus S_\infty(L)} w \otimes (\mathrm{rec}_w(a) - 1)$. Since $\mathrm{rec}_w(a) = 1 \in \Gamma_\chi$ for all $w \in S_\infty(L)$, the proposition follows. □

Proposition 5.15. *Then we have canonical isomorphisms*

$$\ker\beta \simeq H^0(C)_E$$

and

$$\operatorname{coker}\beta \simeq H^1(C)_{\rm tf}\otimes_R P/P^2.$$

Proof. Let δ be the boundary map $H^0(C_E)\to H^1(C\otimes_R P)=H^1(C)\otimes_R P$. We have

$$\ker\delta \simeq \operatorname{coker}(H^0(C\otimes_R P)\to H^0(C))=H^0(C)_E$$

and

$$\operatorname{im}\delta=\ker(H^1(C)\otimes_R P\to H^1(C))=H^1(C)[P]\otimes_R P,$$

where $H^1(C)[P]$ is the submodule of $H^1(C)$ which is annihilated by P. By Proposition 5.12 (iii), we know $H^1(C)[P]=H^1(C)_{\rm tors}$. Hence, the natural map

$$H^1(C)\otimes_R P\to H^1(C)\otimes_R P/P^2\simeq H^1(C)_E\otimes_E P/P^2\simeq H^1(C_E)\otimes_E P/P^2$$

is injective on $H^1(C)_{\rm tors}\otimes_R P$. From this we see that $\ker\beta\simeq H^0(C)_E$. We also have

$$\operatorname{coker}\beta\simeq\operatorname{coker}(H^1(C)_{\rm tors}\otimes_R P\to H^1(C)\otimes_R P/P^2)\simeq H^1(C)_{\rm tf}\otimes_R P/P^2.$$

Hence we have completed the proof. □

By Lemma 5.12, we see that there are canonical isomorphisms

$$H^0(C)_E\simeq U_{K_\infty,S,T}\otimes_\Lambda\mathbb{Q}_p(\chi),$$
$$H^1(C)_E\simeq\mathcal{X}_{K_\infty,S}\otimes_\Lambda\mathbb{Q}_p(\chi),$$
$$H^1(C)_{{\rm tf},E}\simeq\mathcal{Y}_{K_\infty,V}\otimes_\Lambda\mathbb{Q}_p(\chi).$$

Hence, by Proposition 5.15, we have the exact sequence

$$0\to U_{K_\infty,S,T}\otimes_\Lambda\mathbb{Q}_p(\chi)\to e_\chi\mathbb{Q}_p(\chi)U_{L,S,T}$$
$$\xrightarrow{\beta} e_\chi\mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'}\otimes_{\mathbb{Z}_p}I(\Gamma_\chi)/I(\Gamma_\chi)^2)\to\mathcal{Y}_{K_\infty,V}\otimes_\Lambda P/P^2\to 0.$$

This induces an isomorphism

$$\widetilde{\beta}: e_\chi\mathbb{Q}_p(\chi)(\bigwedge^{r'}U_{L,S,T}\otimes\bigwedge^{r'}\mathcal{Y}^*_{L,V'})\xrightarrow{\sim}\bigwedge^{r}(U_{K_\infty,S,T}\otimes_\Lambda\mathbb{Q}_p(\chi))\otimes\bigwedge^{r}(\mathcal{Y}^*_{K_\infty,V}\otimes_\Lambda\mathbb{Q}_p(\chi))\otimes P^e/P^{e+1}.$$

We have isomorphisms

$$\bigwedge^{r'}\mathcal{Y}^*_{L,V'}\xrightarrow{\sim}\mathbb{Z}_p[G_\chi];\ w_1^*\wedge\cdots\wedge w_{r'}^*\mapsto 1,$$

$$\bigwedge^{r}(\mathcal{Y}^*_{K_\infty,V}\otimes_\Lambda\mathbb{Q}_p(\chi))\xrightarrow{\sim}\mathbb{Q}_p(\chi);\ w_1^*\wedge\cdots\wedge w_r^*\mapsto 1.$$

By these isomorphisms, we see that $\widetilde{\beta}$ induces an isomorphism

$$e_\chi\mathbb{Q}_p(\chi)\bigwedge^{r'}U_{L,S,T}\xrightarrow{\sim}\bigwedge^{r}(U_{K_\infty,S,T}\otimes_\Lambda\mathbb{Q}_p(\chi))\otimes P^e/P^{e+1},$$

which we denote also by $\widetilde{\beta}$. Note that we have a natural injection

$$\bigwedge^r (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1} \hookrightarrow e_\chi \mathbb{Q}_p(\chi)(\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e/I(\Gamma_\chi)^{e+1}).$$

Composing this with $\widetilde{\beta}$, we have an injection

$$\widetilde{\beta} : e_\chi \mathbb{Q}_p(\chi) \bigwedge^{r'} U_{L,S,T} \hookrightarrow e_\chi \mathbb{Q}_p(\chi)(\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e/I(\Gamma_\chi)^{e+1}).$$

By Proposition 5.14, we obtain the following

Proposition 5.16. *Let*

$$\mathrm{Rec}_\infty : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p(\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e/I(\Gamma_\chi)^{e+1})$$

be the map defined in §4.1. Then we have

$$(-1)^{re} e_\chi \mathrm{Rec}_\infty = \widetilde{\beta}.$$

In particular, $e_\chi \mathrm{Rec}_\infty$ is injective.

5.3. **The proof of the main result.** In this section we prove Theorem 5.2.

We start with an important technical observation. Let Π_n denote the free $\mathbb{Z}_p[\mathcal{G}_{\chi,n}]$-module $\Pi_{K_\infty} \otimes_\Lambda \mathbb{Z}_p[\mathcal{G}_{\chi,n}]$, and $I(\Gamma_{\chi,n})$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_{\chi,n}]$.

We recall from [9, Lemma 5.19] that the image of

$$\pi^V_{L_n/k,S,T} : \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \to \bigwedge^r \Pi_n$$

is contained in $I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n$ (see Proposition 2.6(iii)) and also from [9, Proposition 4.17] that $\nu_n^{-1} \circ \mathcal{N}_n$ induces the map

$$I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n \to \bigwedge^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e/I(\Gamma_{\chi,n})^{e+1}.$$

Lemma 5.17. *There exists a commutative diagram*

$$\begin{tikzcd}
\det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \arrow[r] \arrow[d, "\pi^V_{L_n/k,S,T}"'] & \det_{\mathbb{Z}_p[G_\chi]}(C_{L,S,T}) \arrow[d, "\pi^{V'}_{L/k,S,T}"] \\
I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n \arrow[d, "\nu_n^{-1}\circ\mathcal{N}_n"'] & \bigcap^{r'} U_{L,S,T} \arrow[d, "(-1)^{re}\mathrm{Rec}_n"] \\
\bigwedge^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e/I(\Gamma_{\chi,n})^{e+1} & \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}} I(\Gamma_{\chi,n})^e/I(\Gamma_{\chi,n})^{e+1}. \arrow[l, hook', "\supset"']
\end{tikzcd}$$

Proof. This follows from Proposition 2.6(iii) and [9, Lemma 5.21]. ∎

For any intermediate field F of K_∞/k, we denote by $\mathcal{L}_{F/k,S,T}$ the image of the (conjectured) element $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ under the isomorphism

$$\mathbb{Z}_p[[\mathrm{Gal}(F/k)]] \otimes_\Lambda \det_\Lambda(C_{K_\infty,S,T}) \simeq \det_{\mathbb{Z}_p[[\mathrm{Gal}(F/k)]]}(C_{F,S,T}).$$

Note that, by the proof of Theorem 3.4, we have

$$\pi^V_{L_n/k,S,T}(\mathcal{L}_{L_n/k,S,T}) = \epsilon^V_{L_n/k,S,T}.$$

Hence, Lemma 5.17 implies that

$$(-1)^{re}\mathrm{Rec}_n(\pi^{V'}_{L/k,S,T}(\mathcal{L}_{L/k,S,T})) = \nu_n^{-1} \circ \mathcal{N}_n(\epsilon^V_{L_n/k,S,T}) =: \kappa_n.$$

We set

$$\kappa := (\kappa_n)_n \in \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_{\chi,n})^e/I(\Gamma_{\chi,n})^{e+1}.$$

Then the validity of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ implies that

$$e_\chi\kappa = (-1)^{re} e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}).$$

In addition, by Proposition 5.16, we know that $e_\chi\mathrm{Rec}_\infty$ is injective, and so

$$\pi^{V'}_{L/k,S,T}(e_\chi\mathcal{L}_{L/k,S,T}) = e_\chi\epsilon^{V'}_{L/k,S,T}.$$

Hence, by Proposition 2.5, we see that Conjecture $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is valid, as claimed.

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